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Random walks in random conductances: Decoupling and spread of infection

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Abstract

Let (G, μ) be a *uniformly elliptic* random conductance graph on $\mathbb{Z}^d$ with a Poisson point process of particles at time $t = 0$ that perform independent simple random walks. We show that inside a cube Q_K of side length K, if all subcubes of side length $\ell < K$ inside Q_K have sufficiently many particles, the particles return to stationarity after $c\ell^2$ time with a probability close to 1. We show that in this setup, an infection spreads with positive speed in any direction. Our framework is robust enough to allow us to also extend the result to infection with recovery.

Keywords: Mixing; Decoupling; Spread of infection; Heat kernel

1. Introduction

We consider the graph $G = (\mathbb{Z}^d, E)$, $d \geq 2$ to be the d-dimensional integer lattice, with edges between nearest neighbors: for $x, y \in \mathbb{Z}^d$ we have $(x, y) \in E$ iff $\|x - y\|_1 = 1$. Let $\{\mu_{x,y}\}_{(x,y)\in E}$ be a collection of i.i.d. non-negative weights, which we call *conductances*. In this paper, conductances will always be symmetric, so $\mu_{x,y} = \mu_{y,x}$ for all $(x, y) \in E$. We also assume

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that the conductances are uniformly elliptic: that is,

$$\text{there exists deterministic } C_M > 0, \text{ such that}$$
$$\mu_{x,y} \in [C_M^{-1}, C_M] \text{ for all } (x, y) \in E,\ \mathbb{P}\text{-}a.s. \tag{1}$$

We say $x \sim y$ if $(x, y) \in E$ and define $\mu_x = \sum_{y \sim x} \mu_{x,y}$. At time 0, consider a Poisson point process of particles on $\mathbb{Z}^d$, with intensity measure $\lambda(x) = \lambda_0 \mu_x$ for some constant $\lambda_0 > 0$ and all $x \in \mathbb{Z}^d$. That is, for each $x \in \mathbb{Z}^d$, the number of particles at x at time 0 is an independent Poisson random variable of mean $\lambda_0 \mu_x$. Then, let the particles perform independent continuous-time simple random walks (CSRW) on the weighted graph so that a particle at $x \in \mathbb{Z}^d$ jumps to a neighbor $y \sim x$ at rate $\frac{\mu_{x,y}}{\mu_x}$. It follows from the thinning property of Poisson random variables that the system of particles is in stationarity; thus, at any time t, the particles are distributed according to a Poisson point process with intensity measure λ.

We study the spread of an infection among the particles. Assume that at time 0 there is at least one particle at the origin,[2] all particles at the origin are infected, and all other particles are uninfected. Then an uninfected particle gets infected as soon as it shares a site with an infected particle. Our first result establishes that the infection spreads with positive speed.

Theorem 1. *Let $\{\mu_{x,y}\}_{(x,y)\in E}$ be i.i.d. satisfying (1). For any time $t \geq 0$, let I_t be the position of the infected particle that is furthest away from the origin. Then*

$$\liminf_{t\to\infty} \frac{\|I_t\|_1}{t} > 0 \quad \textit{almost surely}.$$

The above result has been established on the square lattice (i.e., $\mu_{x,y} = 1$ for all $(x, y) \in E$) by Kesten and Sidoravicius [9] via an intricate multi-scale analysis; see also [10] for a shape theorem. In a companion paper [7], we develop a framework which can be used to analyze processes in this setting without the need of carrying out a multi-scale analysis from scratch. We prove our Theorem 1 via this framework, showing the applicability of our technique from [7]. We also apply this technique to analyze the spread of an infection with recovery. Let the setup be as before, but now each infected particle independently recovers and becomes uninfected at rate γ for some fixed parameter $\gamma > 0$. After recovering, a particle becomes again susceptible to the infection and gets infected again whenever it shares a site with an infected particle. Our next result shows that if γ is small enough, then with positive probability there will be at least one infected particle at all times. When this happens, we also obtain that the infection spreads with positive speed.

Theorem 2. *Let $\{\mu_{x,y}\}_{(x,y)\in E}$ be i.i.d. satisfying (1). For any $\lambda_0 > 0$, there exists $\gamma_0 > 0$ such that, for all $\gamma \in (0, \gamma_0)$, with positive probability, the infection does not die out. Furthermore, there are constants $c_1, c_2, c_3 > 0$ such that*

$$\mathbb{P}[\|I_t\|_1 \geq c_1 t \textit{ for all } t \geq c_3] \geq c_2,$$

where I_t is the position of the infected particle that is furthest away from the origin at time t. We set $I_t = 0$ if the infection dies out before t.

The challenge in this setup comes from the heavily dependent structure of the model. Though particles move independently of one another, dependencies do arise over time. For example, if

[2] We can without loss of generality add an infected particle to the origin, since our results are based on increasing events. See Section 5 for details.

a ball of radius R centered at some vertex x of the graph turns out to have no particles at time 0, then the ball $B(x, R/2)$ of radius $R/2$ centered at x, will continue to be empty of particles up to time R^2, with positive probability. In particular, the probability that the $(d+1)$-dimensional, space–time cylinder $B(x, R/2) \times [0, R^2]$ has no particle is at least $\exp\{-cR^d\}$ for some constant c, which is just a stretched exponential in the volume of the cylinder. On the other hand, one expects that, after time $t \gg R^2$, the set of particles inside the ball will become "close" to stationarity.

To deal with dependences, one often resorts to a decoupling argument, showing that two local events behave roughly independently of each other, provided they are measurable according to regions in space time that are sufficiently far apart. We will obtain such an argument by extending a technique which we call *local mixing*, and which was introduced in [13]. The key observation is the following. Consider a cube $Q \subseteq \mathbb{Z}^d$, tessellated into subcubes of side length $\ell > 0$. For simplicity assume for the moment that $\mu_{x,y} = 1$ for all $(x, y) \in E$. Suppose that at some time t, the configuration of particles inside Q is dense enough, in the sense that inside each subcube there are at least $c\ell^d$ particles, for some constant $c > 0$. Regardless of how the particles are distributed inside Q, as long as the subcubes are dense, we obtain that at some time $t + c'\ell^2$, not only particles had enough time to move out of the subcubes they were in at time t, but also we obtain that the configuration of particles inside "the core" of Q (i.e., away from the boundary of Q) stochastically dominates a Poisson point process of intensity $(1-\epsilon)c\ell^d$ that is independent of the configuration of particles at time t. Moreover, the value ϵ can be made arbitrarily close to 0 by setting c' large enough. In words, we obtain a configuration at time $t + c'\ell^2$ inside the core of Q that is roughly independent of the configuration at time t, and is close to the stationary distribution. To the best of our knowledge, the idea of local mixing in such settings originated in the work of Sinclair and Stauffer [13], and was later applied in [11,14]. This idea was then extended with the introduction of soft local times by Popov and Teixeira [12] (see also [8]), and applied to other processes, such as random interlacements.

Our second main goal in this paper is to show that this local mixing result can be obtained in a larger setting, in which a local CLT, which plays a crucial role in the proof[3] of [13,11,8], might not hold or only holds in the limit as time goes to infinity, with no good control on the convergence rate. This is precisely the situation in our setting, where the weights $\mu_{x,y}$ are not all identical to 1. To work around that, we will show that local mixing can be obtained whenever a so-called *Parabolic Harnack Inequality* holds, and we have some good estimates on the displacement of random walks.

For the result below, we can impose slightly weaker conditions on $\mu_{x,y}$. Let p_c be the critical probability for bond percolation on $\mathbb{Z}^d$. Assume that $\mu_{x,y}$ are i.i.d. and that, for each $(x, y) \in E$, we have

$$\mathbb{P}[\mu_{x,y} = 0] < p_c \text{ and } \mu_{x,y} \text{ satisfies (1) whenever } \mu_{x,y} > 0. \tag{2}$$

For two regions $Q' \subseteq Q \subset \mathbb{Z}^d$, we say that Q' is x away from the boundary of Q if the distance between Q' and Q^c is at least x. We say a cube of side-length a is tessellated into subcubes of side-length b, if a is a multiple of b and the union of the subcubes equals the cube of side-length a.

Theorem 3. *Let $\{\mu_{x,y}\}_{(x,y)\in E}$ be i.i.d. satisfying (2). There exist positive constants c_1, c_2, c_3, c_4, c_5 such that the following holds. Fix $K > \ell > 0$ and $\epsilon \in (0, 1)$. Consider a cube Q of*

[3] The results of [13,11] are in the setting of Brownian motions on $\mathbb{R}^d$, but can be adapted in a straightforward way to random walks on $\mathbb{Z}^d$ with $\mu_{x,y} = 1$ for all $(x, y) \in E$ by using the local CLT.

side-length K, tessellated into subcubes $(T_i)_i$ of side length ℓ. Assume each subcube T_i contains at least $\beta\sum_{x\in T_i}\mu_x$ particles for some $\beta > 0$, and let $\Delta \geq c_1\ell^2\epsilon^{-c_2}$. If ℓ is large enough, then after the particles move for time Δ, we obtain that within a region $Q' \subseteq Q$ that is at least $c_3\ell\epsilon^{-c_4}$ away from the boundary of Q, the particles dominate an independent Poisson point process of intensity measure $\nu(x) = (1-\epsilon)\beta\mu_x$, $x \in Q'$, with probability at least

$$1 - \sum_{y\in Q'} \exp\left\{-c_5\beta\mu_y\epsilon^2\Delta^{d/2}\right\}.$$

We will prove a more detailed version of this theorem in Section 3 (see Theorem 4). Although we only prove the result for the case of conductances on the square lattice, Theorem 3 holds for more general graphs. The theorem holds for any graph G and any region Q of G that can be tessellated into subregions of diameter at most ℓ whenever the particles in each such subregion are dense enough, the so-called parabolic Harnack inequality holds for G and we have estimates on the displacement of random walks on G. We discuss some extensions in Section 4.

The structure of this paper is as follows. In Section 2, we formally define the family of graphs we consider for local mixing and present results concerning the parabolic Harnack inequality, heat kernel bounds and exit times for random walks on such graphs. In Section 3, we state a more precise version of Theorem 3 and prove it. In Section 4 we prove an extension of the local mixing result to random walks whose displacement is conditioned to be bounded, which is particularly useful in applications [13,7]. In Section 5, we use the local mixing result and results from our companion paper [7] to prove Theorems 1 and 2 for graphs satisfying (1).

2. Heat kernel estimates and exit times

In this section, we consider a simple, infinite connected graph $G = (V, E)$, with uniformly bounded degrees. For $x, y \in V$, let $|x-y|$ denote the graph distance between x and y in G. In order to avoid potentially confusing notation, we allow ourselves a slight abuse of notation and also use $|x-y|$ to denote the graph distance when dealing with non-Euclidian graphs. For $x \in V$, let $B(x,r) = \{y \in V : |x-y| \leq r\}$ be the ball of radius r centered at x. We consider non-negative weights (conductances) $(\mu_{x,y})_{(x,y)\in E}$, that are symmetric. As in Section 1, we denote by $x \sim y$ whenever $x, y \in V$ are neighbors in G, and define $\mu_x = \sum_{y\sim x}\mu_{x,y}$. We also extend μ to a measure on V. The reader may think of V as $\mathbb{Z}^d$ and $\mu_{x,y}$ being i.i.d. random variables satisfying (2). We keep our notation in greater generality as we want to highlight the exact conditions we need for our results.

Assume the existence of $d \geq 1$ and C_U such that

$$\mu(B(x,r)) \leq C_U r^d, \quad \text{for all } r \geq 1, \text{ and } x \in V. \tag{3}$$

We consider a continuous time simple random walk on the weighted graph $\mathcal{G} := (G, \mu)$, which jumps from vertex x to vertex y at rate $\frac{\mu_{x,y}}{\mu_x}$ (we consider $\frac{\mu_{x,y}}{\mu_x} = 0$ whenever $\mu_x = 0$). More formally, for any function $f : V \to \mathbb{R}$, let

$$\mathcal{L}f(x) = \mu_x^{-1}\sum_{y\sim x}\mu_{x,y}(f(y) - f(x)), \tag{4}$$

and define the random walk started at vertex x as the Markov process $Y = (Y_t, t \in [0,\infty), \mathbb{P}_x, x \in V)$ with generator $\mathcal{L}$. Its *heat kernel* on the graph is defined as

$$q_t(x,y) = \frac{\mathbb{P}_x(Y_t = y)}{\mu_y}, \text{ for any } x, y \in V. \tag{5}$$

We will say that a particle walks along $\mathcal{G}$ if it is a Markov process with generator $\mathcal{L}$ as defined above. We now state several definitions from [3] which we use throughout the paper.

Definition 1 (*Very Good Balls*). Let C_V, C_P and $C_W \geq 1$ be fixed constants. We say $B(x,r)$ is (C_V, C_P, C_W)-good if:

$$\mu(B(x,r)) \geq C_V r^d,$$

and the weak Poincaré inequality

$$\sum_{y\in B(x,r)} (f(y) - \bar{f}_{B(x,r)})^2 \mu_y \leq C_P r^2 \sum_{y,z\in B(x,C_W r), z\sim y} (f(y) - f(z))^2 \mu_{yz}$$

holds for every $f : B(x, C_W r) \to \mathbb{R}$, where $\bar{f}_{B(x,r)} = \mu(B(x,r))^{-1} \sum_{y\in B(x,r)} f(y)\mu_y$ is the weighted average of f in $B(x,r)$. Furthermore, we say $B(x,R)$ is (C_V, C_P, C_W)-very good if there exists $N_B = N_{B(x,R)} \leq R^{1/(d+2)}$ such that for all $r \geq N_B$, $B(y,r)$ is good whenever $B(y,r) \subseteq B(x,R)$. We assume that $N_B \geq 1$.

For the remainder of the paper we assume that $d \geq 2$, fix C_U, C_V, C_P and C_W and take $\mathcal{G} = ((V,E),\mu)$ to satisfy (3).

We are now ready to present some key results from [4] that control the variation of the random walk density function. We will also present a result about random walk exit times which was initially shown in [3] for Bernoulli percolation clusters and then generalized to our setup in [4]. The first result gives Gaussian upper and lower bounds for the heat kernel for very good balls.

Proposition 1 (*[4, Theorem 2.2]*). *Assume the weights $\mu_{x,y}$ satisfy (1) or (2). Fix a vertex $x \in V$. Suppose there exists $R_1 = R_1(x)$ such that $B(x,R)$ is very good with $N_{B(x,R)}^{3(d+2)} \leq R$ for every $R \geq R_1$. Then there exist positive constants c_1, c_2, c_3, c_4 such that if $t \geq R_1^{2/3}$, we obtain*

$$q_t(x,y) \leq c_1 t^{-d/2} e^{-c_2|x-y|^2/t}, \quad \textit{for all } y \in V \textit{ with } |x-y| \leq t$$

and

$$q_t(x,y) \geq c_3 t^{-d/2} e^{-c_4|x-y|^2/t}, \quad \textit{for all } y \in V \textit{ with } |x-y|^{3/2} \leq t.$$

Now define the space–time regions

$$Q(x,R,T) = B(x,R) \times (0,T],$$
$$Q_-(x,R,T) = B(x,\tfrac{R}{2}) \times [\tfrac{T}{4},\tfrac{T}{2}]$$

and

$$Q_+(x,R,T) = B(x,\tfrac{R}{2}) \times [\tfrac{3T}{4},T].$$

We denote $t + Q(x,R,T) = B(x,R) \times (t,t+T]$ and similarly $t + Q_-(x,R,T) = B(x,\frac{R}{2}) \times [t+\frac{T}{4}, t+\frac{T}{2}]$ and $t + Q_-(x,R,T) = B(x,\frac{R}{2}) \times [t+\frac{3T}{4}, t+T]$. We call a function $u : V \times \mathbb{R} \to \mathbb{R}$ *caloric* on Q if it is defined on $Q = Q(x,R,T)$ and

$$\frac{\partial}{\partial t} u(\hat{x},\hat{t}) = \mathcal{L}u(\hat{x},\hat{t}) \quad \text{for all } (\hat{x},\hat{t}) \in Q.$$

We say the *parabolic Harnack inequality* (PHI) holds with constant C_H for $Q = Q(x,R,T)$ if whenever $u = u(x,t)$ is non-negative and caloric on Q, then

$$\sup_{(\hat{x},\hat{t})\in Q_-(x,R,T)} u(\hat{x},\hat{t}) \leq C_H \inf_{(\hat{x},\hat{t})\in Q_+(x,R,T)} u(\hat{x},\hat{t}).$$

It is well known that the heat kernel of a random walk on $\mathcal{G}$ started at x is a caloric function; in fact taking $\hat{x} = 0$ and $u(x,t) = q_t(0,x)$ we have

$$\begin{aligned}\frac{\partial}{\partial t} q_t(0,x) &= \lim_{dt\to 0} \frac{1}{\mu_x} \frac{\sum_{y\sim x} \mathbb{P}_0(Y_t = y)\mathbb{P}_y(Y_{dt} = x) - \mathbb{P}_0(Y_t = x)\mathbb{P}_x(Y_{dt} \neq x)}{dt} \\ &= \frac{1}{\mu_x}\left(\sum_{y\sim x} \mathbb{P}_0(Y_t = y)\frac{\mu_{y,x}}{\mu_y} - \sum_{y\sim x}\frac{\mu_{x,y}}{\mu_x}\mathbb{P}_0(Y_t = x)\right) \\ &= \frac{1}{\mu_x}\sum_{y\sim x}\mu_{x,y}(q_t(0,y) - q_t(0,x)) = \mathcal{L}q_t(0,x).\end{aligned}$$

The main result from [4] shows that the PHI holds in regions that are very good according to Definition 1.

Proposition 2 (*[4, Theorem 3.1]*). *Assume the weights $\mu_{x,y}$ satisfy (1) or (2). Let $x_0 \in V$. Suppose that $R_1 \geq 16$ and $B(x_0,R_1)$ is (C_V, C_P, C_W)-very good with $N_{B(x_0,R_1)}^{2d+4} \leq R_1/(2\log R_1)$. Then there exists a constant $C_H > 0$ such that the* PHI *holds for $Q(x_1,R,R^2)$ for any $x_1 \in B(x_0,R_1/3)$ and for R such that $R\log R = R_1$.*

A direct consequence of the PHI is the following known proposition, which when applied to the caloric function $u(x,t) = q_t(0,x)$ gives that $q_t(0,x)$ and $q_t(0,y)$ are very similar to each other when x and y are close by. This property will be crucial for our proof of local mixing, so we give the proof of this proposition for completeness.

Proposition 3. *Assume the weights $\mu_{x,y}$ satisfy (1) or (2). Let $x_0 \in V$. Suppose that there exists $s(x_0) \geq 0$ so that for all $R \geq s(x_0)$, the* PHI *holds with constant $C_H > 1$ for $Q(x_0,R,R^2)$. Let $\Theta = \log_2(C_H/(C_H - 1))$, and for $x, y \in V$ define*

$$\rho(x_0,x,y) = s(x_0) \vee |x_0 - x| \vee |x_0 - y|.$$

There exists a constant $c > 0$ such that the following holds. Let $r_0 \geq 2s(x_0)$ and suppose that $u = u(x,t)$ is caloric in $Q = Q(x_0,r_0,r_0^2)$. Then for any $x_1, x_2 \in B(x_0,\frac{1}{2}r_0)$ and any t_1,t_2 such that $r_0^2 - \rho(x_0,x_1,x_2)^2 \leq t_1,t_2 \leq r_0^2$ we have

$$|u(x_1,t_1) - u(x_2,t_2)| \leq c\left(\frac{\rho(x_0,x_1,x_2)}{r_0}\right)^{\Theta} \sup_{(t,x)\in Q_+(x_0,r_0,r_0^2)} |u(t,x)|. \tag{6}$$

Proof. For any integer $k \geq 0$, set $r_k = 2^{-k}r_0$, and let

$$\begin{aligned}Q(k) &= (r_0^2 - r_k^2) + Q(x_0,r_k,r_k^2),\\ Q_+(k) &= (r_0^2 - r_k^2) + Q_+(x_0,r_k,r_k^2)\end{aligned}$$

and

$$Q_-(k) = (r_0^2 - r_k^2) + Q_-(x_0,r_k,r_k^2).$$

This gives that $Q_+(k) = Q(k+1)$. Now take $k \geq 1$ small enough, so that $r_k \geq 2s(x_0)$. If we apply the PHI to the non-negative caloric functions $-u + \sup_{Q(k)} u$ and $u - \inf_{Q(k)} u$, we get the inequalities

$$\sup_{Q(k)} u - \inf_{Q_-(k)} u \leq C_H(\sup_{Q(k)} u - \sup_{Q_+(k)} u)$$

and

$$\sup_{Q_-(k)} u - \inf_{Q(k)} u \le C_H\big(\inf_{Q_+(k)} u - \inf_{Q(k)} u\big).$$

Adding them together and using $\sup_{Q_-(k)} u - \inf_{Q_-(k)} u \ge 0$ gives

$$\sup_{Q(k)} u - \inf_{Q(k)} u \le C_H\big(\sup_{Q(k)} u - \inf_{Q(k)} u\big) - C_H\big(\sup_{Q_+(k)} u - \inf_{Q_+(k)} u\big).$$

Denoting by $\mathrm{Osc}(u, A) = \sup_A u - \inf_A u$ and setting $\delta = C_H^{-1}$, this gives

$$\mathrm{Osc}(u, Q_+(k)) \le (1-\delta)\,\mathrm{Osc}(u, Q(k)). \tag{7}$$

Next, take the largest m such that $r_m \ge \rho(x_0, x_1, x_2)$. Then, applying (7) repeatedly on $Q(1) \supset Q(2) \supset \dots Q(m)$ yields, since $(x_i, t_i) \in Q(m)$,

$$|u(t_1, x_1) - u(t_2, x_2)| \le \mathrm{Osc}(u, Q(m)) \le (1-\delta)^{m-1}\,\mathrm{Osc}(u, Q(1)).$$

Since

$$(1-\delta)^m = 2^{-m\Theta} \le \left(\frac{2\rho(x_0, x_1, x_2)}{r_0}\right)^{\Theta},$$

the result follows. □

We will also need to control the exit time of the random walk out of a ball of radius r, which we define as

$$\tau(x, r) = \inf\{t : Y_t \notin B(x, r)\}.$$

Proposition 4. *Assume the weights $\mu_{x,y}$ satisfy (1) or (2). Let $x_0 \in V$ and let $B(x_0, R)$ be (C_V, C_P, C_W)-very good with $N_B^{d+2} < R$. Let $x \in B(x_0, \frac{5}{9}R)$. There exist positive constants c_1, c_2, c_3, c_4 such that if t, r satisfy*

$$0 < r \le R \quad \text{and} \quad c_1 N_B^d (\log N_B)^{1/2} r \le t \le c_2 R^2/\log R, \tag{8}$$

then we have

$$\mathbb{P}_x(\tau(x, r) < t) \le c_3 \exp\{-c_4 r^2/t\}. \tag{9}$$

Proof. The proposition was proven for percolation clusters in [3, Proposition 3.7]. The proof for more general $\mathcal{G}$ is similar and can be found in [4, Theorem 2.2a]. □

Since Propositions 1, 2 and 4 rely on very good balls and the related value N_B, we can assume a lower bound S such that if $R > S$, then the conditions of all three are satisfied. More formally, we assume the following.

Assumption 1. The graph G has polynomial growth; i.e., it satisfies (3). Furthermore, there exists a sufficiently large valued positive function $S : V \mapsto \mathbb{R}$ such that for all $x_0 \in V$ and all R_1 with $R_1 \log R_1 \ge S(x_0)$, the ball $B(x_0, R_1)$ is (C_V, C_P, C_W)-very good with $N_{B(x_0,R_1)}^{2d+4} \le R_1$. As a consequence, Propositions 1–4 all hold for any $R > S(x_0)$.

For i.i.d. weights as defined in Section 1, we obtain the following.

Proposition 5. *If $V = \mathbb{Z}^d$ and the weights $\mu_{x,y}$ are i.i.d. and satisfy (1) or (2), then Assumption 1 holds. Furthermore, we have that there exist constants $c, \gamma > 0$ such that*

$$\mathbb{P}[S(x) \geq n] \leq c \exp\{-cn^{\gamma}\} \text{ for all } x \in \mathbb{Z}^d \text{ and } n \geq 0.$$

If the weights $\mu_{x,y}$ are i.i.d. and satisfy (1), then Assumption 1 holds with $S(x) = 1$ for all $x \in V$.

Proof. When the weights $\mu_{x,y}$ satisfy (1) (i.e. are bounded away from 0 and infinity), Delmotte [6] has shown that the heat kernel bounds from Proposition 1 and the PHI from Proposition 2 hold for all balls $B(x, R)$, for any R and all x. Therefore, by [4, Theorem 5.7] we can set $S(x) \equiv 1$.

For the case when $\mu_{x,y}$ satisfy (2), we consider first the case when $\mu_{x,y} \in \{0, 1\}$. Then, [3, Theorem 1] gives the existence of the heat kernel bounds for $t \geq S(x)$ and [3, Lemma 2.19] gives the required bound on its tail. [4, Theorem 2.2] then generalizes this result for weights that satisfy (2) and proves the validity of Proposition 2, for weights satisfying either (1) or (2). □

Remark 1. In [5] it has been shown that when the weights $\mu_{x,y}$ are i.i.d. but can assume values arbitrarily close to zero, so neither (1) nor (2) hold, it is possible to find distributions (at least in dimensions $d \geq 5$) for which Assumption 1 does not hold. Hence, even though we do not explicitly use uniform ellipticity of $\mu_{x,y}$ in our proofs, this property has a fundamental role in our analysis. Recent results (see, for example, [1]) have been derived to relax assumption (2), but they do not establish all the properties we need.

Remark 2. In this paper we limit ourselves to the so-called constant speed random walk (CSRW). Similar results to the ones listed in this section also exist for variable speed random walks (VSRW), i.e. random walks where the jump rate from site x to site y is $\mu_{x,y}$ instead of $\frac{\mu_{x,y}}{\mu_x}$ (see for example [1,2]). Similar to the previous remark, these results do not imply all the properties we need, though we believe that with some additional assumptions our approach can be applied also to the VSRW case.

3. Decoupling via local mixing

In this section, we will restrict to the case $V = \mathbb{Z}^d$ and $(x, y) \in E$ if and only if $\|x - y\|_1 = 1$, but we do not assume the $\mu_{x,y}$ are i.i.d. We define a cube of side length $z > 0$ as $Q_z := [-z/2, z/2]^d$. In the remainder of the paper, we will work with the heat kernel q_t as defined in (5). Since we allow $\mu_{x,y} = 0$, it is possible for two sites not to be connected. To address this we require the existence of an infinite component. Formally, we assume the following.

Assumption 2. For each $(x, y) \in E$, either $\mu_{x,y} = 0$ or it satisfies (1) for a uniform constant C_M. Moreover, the weights $\mu_{x,y}$ are such that an infinite connected component of edges of positive weight within $\mathcal{G}$ exists and contains the origin.

With this let $\mathcal{C}_\infty$ be the infinite connected component of $\mathcal{G}$ that contains the origin and define

$$\tilde{Q}_z := Q_z \cap \mathcal{C}_\infty.$$

We note that if $\mu_{x,y}$ satisfy (1), then Assumption 2 is automatically satisfied. If instead (2) holds We will continue to call $\tilde{Q}_z$ as a "cube". We are now ready to state the more detailed version of Theorem 3.

Theorem 4. *Let $\mu_{x,y}$ satisfy Assumptions 1 and 2 and let $c > 0$ be an arbitrary constant. There exist constants c_0, c_1, $C > 0$ such that the following holds. Fix large enough $K > \ell > 0$, $\epsilon \in (0, 1)$. Consider the cube Q_K tessellated into subcubes $(T_i)_i$ of side length ℓ. Let $(x_j)_j \subset \tilde{Q}_K$ be the locations at time 0 of a collection of particles, such that each subcube $\tilde{T}_i$ contains at least $\sum_{y\in\tilde{T}_i}\beta\mu_y$ particles for some $\beta > 0$. Assume that $\ell > S^{d+1}(x)$ for all $x \in \tilde{Q}_K$ and sufficiently large so that $\sum_{y\in\tilde{T}_i}\beta\mu_y \geq c$ for all subcubes $\tilde{T}_i$. Let $\Delta \geq c_0\ell^2\epsilon^{-4/\Theta}$ where Θ is as in Proposition 3. For each j denote by Y_j the location of the jth particle at time Δ. Fix $K' > 0$ such that $K - K' \geq \sqrt{\Delta}c_1\epsilon^{-1/d}$. Then there exists a coupling $\mathbb{Q}$ of an independent Poisson point process ψ with intensity measure $\zeta(y) = \beta(1-\epsilon)\mu_y$, $y \in \mathcal{C}_\infty$, and $(Y_j)_j$ such that within $\tilde{Q}_{K'} \subset \tilde{Q}_K$, ψ is a subset of $(Y_j)_j$ with probability at least*

$$1 - \sum_{y\in\tilde{Q}_{K'}} \exp\left\{-C\beta\mu_y\epsilon^2\Delta^{d/2}\right\}.$$

Note that, due to Proposition 5, Theorem 3 is a special case of Theorem 4, which we prove below. In order to do so, we will use something called *soft local times*, which was introduced in [12] to analyze random interlacements, following the introduction of local mixing in [13,11,14]; see also [8] for an application of this technique to random walks on $\mathbb{Z}^d$.

Proposition 6. *Let $(Z_j)_{j\leq J}$ be a collection of J independent random particles on V distributed according to a family of density functions $g_j : V \to \mathbb{R}$, $j \leq J$. Define for all $y \in V$ the soft local time function $H_J(y) = \sum_{j=1}^J \xi_j g_j(y)$, where the ξ_j are i.i.d. exponential random variables of mean 1. Let ψ be a Poisson point process on V with intensity measure $\rho : V \to \mathbb{R}$ and define the event $E = \left\{\text{the particles belonging to } \psi \text{ are a subset of } (Z_j)_{j\leq J}\right\}$. Then there exists a coupling between $(Z_j)_{j\leq J}$ and ψ, such that*

$$\mathbb{P}[E] \geq \mathbb{P}[H_J(y) \geq \rho(y),\ \forall y \in V].$$

Proof. The coupling is introduced in [12, Section 4] and proven in [12, Corollary 4.4]. A reformulation of the construction for particles on a graph can be found in [8, Appendix A], and our claim corresponds to [8, Corollary A.3]. □

We are now ready to prove Theorem 4.

Proof of Theorem 4. By Proposition 6, there exists a coupling $\mathbb{Q}$ of an independent Poisson point process ψ with intensity measure $\zeta(y) = \beta(1-\epsilon)\mu_y\mathbb{1}_{\{y\in\tilde{Q}_{K'}\}}$ and the locations of the particles Y_j, which are distributed according to the density functions $f_\Delta(x_j, y) := q_\Delta(x_j, y)\mu_y$, $y \in \mathcal{C}_\infty$, such that the particles belonging to ψ are a subset of $(Y_j)_j$ with probability at least

$$\mathbb{Q}[H_J(y) \geq \beta\mu_y(1-\epsilon),\ \forall y \in \tilde{Q}_{K'}],$$

where $H_J(y) = \sum_{j=1}^J \xi_j f_\Delta(x_j, y)$, $(\xi_j)_{j\leq J}$ are i.i.d. exponential random variables with parameter 1, and J is the number of particles inside $\tilde{Q}_K$. We first observe that the probability of the converse event is

$$\begin{aligned}\mathbb{Q}[\exists y \in \tilde{Q}_{K'} :\ H_J(y) < \beta\mu_y(1-\epsilon)] &\leq \sum_{y\in\tilde{Q}_{K'}} \mathbb{Q}[H_J(y) < \beta\mu_y(1-\epsilon)]\\ &\leq \sum_{y\in\tilde{Q}_{K'}} e^{\kappa\mu_y\beta(1-\epsilon)}\mathbb{E}^{\mathbb{Q}}[\exp\{-\kappa H_J(y)\}],\end{aligned}$$

where we used Markov's inequality in the last step, which is valid for any $\kappa > 0$. Let c_1 be a large positive constant which we will fix later and let

$$R = \sqrt{\Delta} c_1 \epsilon^{-1/d}.$$

Let J' be a subset of $\{1, 2, \ldots, J\}$ such that for each $\tilde{T}_i$, J' contains exactly $\lceil \sum_{y \in \tilde{T}_i} \beta \mu_y \rceil$ particles that are inside $\tilde{T}_i$. Define $J'(y) \subseteq J'$ to be the set of $j \in J'$ such that $|x_j - y| \leq R$ and define $H'(y)$ as $H_J(y)$ but with the sum restricted to $j \in J'(y)$. Since $H_J(y) \geq H'(y)$ we get that

$$\mathbb{E}^{\mathbb{Q}}[\exp\{-\kappa H_J(y)\}] \leq \mathbb{E}^{\mathbb{Q}}[\exp\{-\kappa H'(y)\}]. \tag{10}$$

Next, we use that the ξ_j in the definition of H are independent exponential random variables to obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\exp\{-\kappa H'(y)\}] &= \prod_{j \in J'(y)} \mathbb{E}^{\mathbb{Q}}[\exp\{-\kappa \xi_j f_\Delta(x_j, y)\}] \\ &= \prod_{j \in J'(y)} \left(1 + \kappa f_\Delta(x_j, y)\right)^{-1}. \end{aligned} \tag{11}$$

Using Taylor's expansion we have that $\log(1 + x) \geq x - x^2$ for $|x| \leq \frac{1}{2}$. Since $\ell \geq S(x)$, we have for all x for which $|x - y| \leq R + \sqrt{d}\ell$ that Proposition 1 holds, and so $q_\Delta(x, y) \leq c_2 \Delta^{-d/2}$ for a constant $c_2 > 0$ and all $y \in \tilde{Q}_{K'}$ and all $x \in \bigcup \tilde{T}_i$, where the union runs across all $\tilde{T}_i$ for which there exists $j \in J'(y)$ such that $x_j \in \tilde{T}_i$. Note that by definition, making the constant c_0 large ensures for $\ell \geq S(x)$ that Δ is large enough with respect to Proposition 1. Hence if $\kappa = C\epsilon\Delta^{d/2}$ for the constant $C = (4C_U c_2)^{-1}$, then

$$\sup_{x \in B(y, R + \sqrt{d}\ell)} \kappa f_\Delta(x, y) = \sup_{x \in B(y, R + \sqrt{d}\ell)} \kappa \mu_y q_\Delta(x, y) \leq C_U c_2 \kappa \Delta^{-d/2} < \frac{\epsilon}{4}.$$

For such a value of κ we have

$$\begin{aligned} \prod_{j \in J'(y)} \left(1 + \kappa f_\Delta(x_j, y)\right)^{-1} &\leq \prod_{j \in J'(y)} \exp\left\{-\kappa f_\Delta(x_j, y)(1 - \kappa f_\Delta(x_j, y))\right\} \\ &\leq \exp\left\{-\sum_{j \in J'(y)} \kappa f_\Delta(x_j, y) \left(1 - \sup_{x \in B(y, R + \sqrt{d}\ell)} \kappa f_\Delta(x, y)\right)\right\} \\ &\leq \exp\left\{-\kappa \sum_{j \in J'(y)} f_\Delta(x_j, y)(1 - \epsilon/4)\right\}. \end{aligned} \tag{12}$$

We claim that

$$\sum_{j \in J'(y)} f_\Delta(x_j, y) \geq \beta \mu_y (1 - \epsilon/2), \tag{13}$$

which together with (12), (11) and (10) give that

$$\begin{aligned} \mathbb{Q}\left[\exists y \in \tilde{Q}_{K'} : \ H_J(y) < \beta \mu_y (1 - \epsilon)\right] &\leq \exp\left\{\kappa \mu_y \beta (1 - \epsilon) - \kappa \beta \mu_y (1 - \epsilon/2)(1 - \epsilon/4)\right\} \\ &\leq \exp\left\{-\kappa \beta \mu_y \epsilon/4\right\}. \end{aligned}$$

Using the value of κ gives the theorem.

It remains to show (13). For each $\tilde{T}_i$ and each particle $x_j \in \tilde{T}_i$, let $x'_j \in \tilde{T}_i$ be such that $f_\Delta(x'_j, y) = \max_{w\in\tilde{T}_i} f_\Delta(w, y)$. Then, write

$$\sum_{j\in J'(y)} f_\Delta(x_j, y) \geq \sum_{j\in J'(y)} \left(f_\Delta(x'_j, y) - |f_\Delta(x'_j, y) - f_\Delta(x_j, y)| \right). \tag{14}$$

We have for each $\tilde{T}_i$

$$\begin{aligned}\sum_{\substack{j\in J'(y)\\ x_j\in\tilde{T}_i}} f_\Delta(x'_j, y) &= \max_{w\in\tilde{T}_i} f_\Delta(w, y) \sum_{\substack{j\in J'(y)\\ x_j\in\tilde{T}_i}} 1 \\ &\geq \max_{w\in\tilde{T}_i} f_\Delta(w, y) \sum_{z\in\tilde{T}_i} \beta\mu_z \\ &\geq \sum_{z\in\tilde{T}_i} \beta\mu_z f_\Delta(z, y). \end{aligned} \tag{15}$$

Set $R(y)$ to be the set of all sites z such that $|z - y| \leq R - \sqrt{d}\ell$; the right hand side of this expression is positive since by definition R is proportional to ℓ and c_1 is assumed to be large. Note that if $z \in R(y)$ then for all particles x_j with $x'_j = z$ and $j \in J'$ we have $j \in J'(y)$. We observe that since $\mu_z f_\Delta(z, y) = \mu_y f_\Delta(y, z)$, we have by using (15) for each $\tilde{T}_i$ that

$$\begin{aligned}\sum_{j\in J'(y)} f_\Delta(x'_j, y) &\geq \sum_{z\in R(y)} \beta\mu_z f_\Delta(z, y) \\ &= \beta\mu_y \sum_{z\in R(y)} f_\Delta(y, z).\end{aligned}$$

Then, since $\ell > S^{d+1}(x)$ we have by Proposition 4 that there exist constants c_4 and c_5 such that

$$\begin{aligned}\sum_{j\in J'(y)} f_\Delta(x'_j, y) &\geq \beta\mu_y \mathbb{P}_y(\tau(y, R - \sqrt{d}\ell) \geq \Delta) \\ &\geq \beta\mu_y(1 - c_4 \exp\{-c_5 c_1^2 \epsilon^{-2/d}\}) \\ &\geq \beta\mu_y(1 - \epsilon/4),\end{aligned} \tag{16}$$

where we set c_1 large enough with respect to c_4 and c_5 for the last inequality to hold.

Now it remains to obtain an upper bound for the term $\sum_{j\in J'(y)} |f_\Delta(x'_j, y) - f_\Delta(x_j, y)|$. We define I to be the set of all i such that $\tilde{T}_i$ contains a particle x_j from the set $(x_j)_{j\in J'(y)}$. Then, since $\ell > S(x)$, there exist positive constants C_{PHI} and C_{BH} such that if we apply the PHI (cf. Proposition 3) with

$$r_0^2 = \Delta \geq c_0 \ell^2 \epsilon^{-4/\Theta} \tag{17}$$

for some constant $c_0 > d$, we obtain

$$\begin{aligned}\sum_{j\in J'(y)} |f_\Delta(x'_j, y) - f_\Delta(x_j, y)| &= \sum_{i\in I} \sum_{\substack{j\in J'(y):\\ x_j\in\tilde{T}_i}} |f_\Delta(x'_j, y) - f_\Delta(x_j, y)| \\ &= \mu_y \sum_{i\in I} \sum_{\substack{j\in J'(y):\\ x_j\in\tilde{T}_i}} |q_\Delta(x'_j, y) - q_\Delta(x_j, y)|\end{aligned}$$

$$\leq \mu_y \sum_{i \in I} \sum_{\substack{j \in J'(y): \\ x_j \in \tilde{T}_i}} \frac{C_{PHI} \ell^{\Theta}}{\Delta^{\Theta/2}} C_{BH} \Delta^{-d/2}$$

$$\leq \mu_y \sum_{i \in I} \sum_{x \in \tilde{T}_i} \frac{2 \max\left\{1, \frac{1}{c}\right\} \beta \mu_x C_{PHI} \ell^{\Theta}}{\Delta^{\Theta/2}} C_{BH} \Delta^{-d/2},$$

where in the first inequality we replaced the supremum term coming from Proposition 3 by its upper bound $C_{BH}\Delta^{-d/2}$ from Proposition 1, and used that $r_0 = \sqrt{\Delta}$ in the bound from Proposition 3. Then

$$\sum_{j \in J'(y)} |f_{\Delta}(x'_j, y) - f_{\Delta}(x_j, y)| \leq 2 \max\left\{1, \frac{1}{c}\right\} \beta \mu_y C_{PHI} C_{BH} \sum_{i \in I} \sum_{x \in \tilde{T}_i} \mu_x \ell^{\Theta} \Delta^{-(d+\Theta)/2}$$

$$\leq 2 \max\left\{1, \frac{1}{c}\right\} \beta \mu_y C_{PHI} C_{BH} C_U R^d \ell^{\Theta} \Delta^{-(d+\Theta)/2}$$

$$\leq \beta \mu_y \frac{\epsilon}{4}, \tag{18}$$

where the last inequality holds by using $\Delta \geq c_0 \ell^2 \epsilon^{-4/\Theta}$ and setting $c_0 > (2 \max\left\{1, \frac{1}{c}\right\} C_{PHI} C_{BH} C_U c_1^d)^{-2/\theta}$. Note that in order to use Proposition 3, we need to have that each pair x_j, x'_j is contained in some ball $B(x_0, r_0/2)$. This is satisfied since $\|x_j - x'_j\| \leq \sqrt{d}\ell$ and r_0 is set sufficiently large by (17). Plugging (18) and (16) into (14) proves (13). □

4. Extensions

Although the estimate derived in Theorem 4 does not depend on the particles outside of Q_K at time 0 when $K - K'$ is sufficiently large, it still depends on the geometry of the entire graph outside of Q_K. In some applications, as in our companion paper [7], one needs to apply this coupling in many different regions of the graph simultaneously. In such cases, in order to control dependences between different regions, it is important that the coupling procedure depends only on the local structure of the graph. In order to do this, we will condition the particles to be inside some large enough, but finite region while they move for time Δ. Recall that, for any $\rho > 0$, $Q_\rho = [-\rho/2, \rho/2]^d$ is the cube of side length ρ. For any $\rho > 0$, we say that a random walk has *displacement* in Q_ρ during $[0, \Delta]$ if the random walk never exits $x + Q_\rho$ during the time interval $[0, \Delta]$, where x is the starting vertex of the random walk.

Lemma 1. *Let $\mu_{x,y}$ satisfy* Assumptions 1 and 2. *There exist constants c_1 and c_2 so that the following holds. Let $V = \mathbb{Z}^d$, $\ell > 0$ and consider the cube $\tilde{Q}_\ell$. Assume $\ell > S(x)$ for all $x \in \tilde{Q}_\ell$. Let $\Delta > c_1\ell^2$ and $\rho \geq c_2\sqrt{\Delta \log \Delta}$. Consider a random walk Y that moves along $\mathcal{G}$ for time Δ conditioned on having its displacement in Q_ρ during the time interval $[0, \Delta]$. Let $x, y \in \tilde{Q}_\ell$ with x being the starting point of the walk, and define*

$$g(x, y) := \mathbb{P}_x \left[Y_\Delta = y \mid Y \text{ has displacement in } Q_\rho \text{ during } [0, \Delta] \right].$$

Then there exists a constant $C > 2$ such that for $x, y, z \in \tilde{Q}_\ell$ we have

$$\left| \frac{g(x, y)}{\mu_y} - \frac{g(z, y)}{\mu_y} \right| \leq C \ell^{\Theta} \Delta^{-(d+\Theta)/2}.$$

Remark 3. Note that the above bound has the same form as the one for the heat kernel of unconditioned random walks in Proposition 3, with the supremum being bounded above by the heat kernel bound from Proposition 1. This allows us to extend Theorem 4 to random walks conditioned to have a bounded displacement during $[0, \Delta]$.

Proof of Lemma 1. Denote by $p_E(\rho)$ the probability that a random walk started at x has displacement in Q_ρ during $[0, \Delta]$. From Proposition 4 , we have that if Δ is sufficiently big, then

$$\begin{aligned} 1 - p_E(\rho) &\leq \mathbb{P}_x[Y \text{ exits } B(x, \rho/2) \text{ during } [0, \Delta]] \\ &= \mathbb{P}_x(\tau(x, \rho/2) < \Delta) \\ &\leq c_a \exp\{-c_b \rho^2/\Delta\}. \end{aligned} \tag{19}$$

Next, using $h(x, y) := \mathbb{P}_x\left[Y_\Delta = y \mid Y \text{ exits } x + Q_\rho \text{ during } [0, \Delta]\right]$ and $f_\Delta(x, y) = \mathbb{P}_x[Y_\Delta = y]$, we can write

$$f_\Delta(x, y) = g(x, y) p_E(\rho) + h(x, y)(1 - p_E(\rho)).$$

With this we have

$$g(x, y) \leq f_\Delta(x, y) \frac{1}{p_E(\rho)}. \tag{20}$$

Then, we can write

$$\begin{aligned} \left| \frac{g(x, y)}{\mu_y} - \frac{g(z, y)}{\mu_y} \right| &= \mathbb{1}_{\{g(x,y)>g(z,y)\}} \left(\frac{g(x, y)}{\mu_y} - \frac{g(z, y)}{\mu_y} \right) \\ &\quad + \mathbb{1}_{\{g(x,y)<g(z,y)\}} \left(\frac{g(z, y)}{\mu_y} - \frac{g(x, y)}{\mu_y} \right) \\ &\leq \mathbb{1}_{\{g(x,y)>g(z,y)\}} \left(\frac{f_\Delta(x, y)}{\mu_y p_E(\rho)} - \frac{f_\Delta(z, y)}{\mu_y p_E(\rho)} + \frac{h(z, y)(1 - p_E(\rho))}{p_E(\rho)\mu_y} \right) \\ &\quad + \mathbb{1}_{\{g(x,y)<g(z,y)\}} \left(\frac{f_\Delta(z, y)}{\mu_y p_E(\rho)} - \frac{f_\Delta(x, y)}{\mu_y p_E(\rho)} + \frac{h(x, y)(1 - p_E(\rho))}{p_E(\rho)\mu_y} \right) \\ &\leq \frac{|q_\Delta(y, x) - q_\Delta(y, z)|}{p_E(\rho)} + \frac{\max\{h(x, y), h(z, y)\}(1 - p_E(\rho))}{p_E(\rho)\mu_y}. \end{aligned}$$

Note that $h(x, y)$ can be written as $\mathbb{E}[f_{\Delta-\tau}(w, y) \mid \tau < \Delta]$, where τ is the first time Y exists $x + Q_\rho$ and w is the random vertex at the boundary of $x + Q_\rho$ where Y is at time τ. Since the weights $\mu_{x,y}$ satisfy (3) by Assumption 1, we have that $\frac{f_{\Delta-\tau}(w,y)}{\mu_y}$ is at most some positive constant c. This holds because either $\Delta - \tau$ is larger than $|w - y|$, which allows us to apply heat kernel bounds from Proposition 1, or $\Delta - \tau$ is smaller than $|w - y|$ so $f_{\Delta-\tau}(w, y)$ is bounded above by the probability that a random walk jumps at least $|w - y|$ steps in time $\Delta - \tau$, which is small enough since $|w - y|$ is large. This gives that $\frac{\max\{h(x,y),h(z,y)\}}{\mu_y}$ is at most c. With this and (19) we obtain that

$$\begin{aligned} \frac{\max\{h(x, y), h(z, y)\}(1 - p_E(\rho))}{\mu_y p_E(\rho)} &\leq \frac{c c_a}{p_E(\rho)} \exp\left\{ \frac{-c_b \rho^2}{\Delta} \right\} \\ &\leq \frac{c c_a}{p_E(\rho)} \exp\{-c_b c_2 \log \Delta\}. \end{aligned}$$

By (19) we can just bound $p_E(\rho)$ below by $1/2$. Then, applying Proposition 5 to $|q_\Delta(y,x) - q_\Delta(y,z)|$, and using Proposition 1 to bound the resulting supremum term, concludes the proof. □

The next theorem is an adaptation of Theorem 4 for conditioned random walks. Note that we need a stronger condition on $K - K'$ below than in Theorem 4.

Theorem 5. *Let $\mu_{x,y}$ satisfy* Assumptions 1 and 2 *and let $c > 0$ be an arbitrary constant. There exist constants c_0, c_1, $C > 0$ such that the following holds. Fix large enough $K > \ell > 0$ and $\epsilon \in (0,1)$. Consider the cube Q_K tessellated into subcubes $(T_i)_i$ of side length ℓ. Let $(x_j)_j \subset \tilde{Q}_K$ be the locations at time 0 of a collection of particles, such that each subcube $\tilde{T}_i$ contains at least $\sum_{y\in\tilde{T}_i}\beta\mu_y$ particles for some $\beta > 0$. Assume that $\ell > S^{d+1}(x)$ for all $x \in \tilde{Q}_K$ and sufficiently large so that $\sum_{y\in\tilde{T}_i}\beta\mu_y \geq c$ for all subcubes $\tilde{T}_i$. Let $\Delta \geq c_0\ell^2\epsilon^{-4/\Theta}$, where Θ is as in Proposition 3. Fix $K' > 0$ such that $K - K' \geq c_1\sqrt{\Delta\log\Delta}$. For each j, denote by Y_j the location of the jth particle at time Δ, conditioned on having displacement in $Q_{K-K'}$ during $[0,\Delta]$. Then there exists a coupling $\mathbb{Q}$ of an independent Poisson point process ψ with intensity measure $\zeta(y) = \beta(1-\epsilon)\mu_y$, $y \in \tilde{Q}_K$, and $(Y_j)_j$ such that within $\tilde{Q}_{K'} \subset \tilde{Q}_K$, ψ is a subset of $(Y_j)_j$ with probability at least*

$$1 - \sum_{y\in\tilde{Q}_{K'}} \exp\left\{-C\beta\mu_y\epsilon^2\Delta^{d/2}\right\}.$$

Proof. Using Lemma 1 and (20) when setting κ, the proof goes in the same way as the proof of Theorem 4. The independence from G outside of $\tilde{Q}_K$ follows from the fact that we only consider particles which have displacement in $Q_{K-K'}$ and ended in $\tilde{Q}_{K'}$, so that they never left $\tilde{Q}_K$ during $[0,\Delta]$. □

4.1. Extension to other graphs

We have shown that the local mixing result of Theorems 4 and 5 works for $\mathbb{Z}^d$, but they can easily be extended to the more general graphs defined in Section 2, as long as Assumptions 1 and 2 hold.

We start with a region $A \subseteq \mathcal{C}_\infty$ around the origin of G and tessellate it into tiles $(T_i)_{i\in I}$ of diameter at most ℓ. Let Δ be as in Theorem 4. Let $A' \subset A$ be all the sites in A that are at least $\sqrt{\Delta}c_1\epsilon^{-1/d} + c\ell$ away from the boundary of A. Then, if A' is not empty, using the same steps as in the proof of Theorem 4, if each tile T_i of A contains at least $\beta\sum_{y\in T_i}\mu_y$ particles at time 0, it holds that in the region A', there is a coupling with an independent Poisson point process ψ of intensity measure $\zeta(y) = \beta(1-\epsilon)\mu_y$ such that at time Δ the particles inside A' are contained in ψ with probability at least

$$1 - \sum_{y\in A'} \exp\left\{-C\beta\mu_y\epsilon^2\Delta^{d/2}\right\},$$

for some constant $C > 0$.

Furthermore, Theorem 5 can analogously be extended in the same way, if we require that A' contains only sites that are at least $c_1\sqrt{\Delta\log\Delta}$ away from the boundary of A, for some constant c_1, and if we condition the random walks to have their displacement limited to a ball of radius $c_1\sqrt{\Delta\log\Delta}$.

5. Spread of the infection

Our goal in this section will be to use Theorem 5 in order to show that on the graph $G = (V, E)$ with $V = \mathbb{Z}^d$ and $E = \{(x, y) : \|x - y\|_1 = 1\}$, and with $\mu_{x,y}$, $(x, y) \in E$ being i.i.d. and satisfying (1), information spreads with positive speed in any direction, as claimed in Theorems 1 and 2. In this setting, Proposition 5 guarantees that Assumption 1 holds with $S(x) \equiv 1$ and since $\mu_{x,y} \neq 0$ for all $(x, y) \in E$, we also have that Assumption 2 holds.

Recall that we assume $d \geq 2$. Tessellate $\mathbb{Z}^d$ into cubes of side length ℓ, indexed by $i \in \mathbb{Z}^d$. Next, tessellate time into intervals of length β, indexed by $\tau \in \mathbb{Z}$. With this we denote by the space–time cell $(i, \tau) \in \mathbb{Z}^{d+1}$ the region $\prod_{j=1}^{d}[i_j\ell, (i_j + 1)\ell] \times [\tau\beta, (\tau + 1)\beta]$. In the following, β is set as a function of ℓ so that the ratio β/ℓ^2 is fixed first to be a small constant, and then ℓ is set sufficiently large.

We will use a result from [7] that gives the existence of a Lipschitz connected surface (cf. Definitions 3 and 4) that surrounds the origin and which is composed of space–time cells, for which a certain local event holds. This will allow us to obtain an infinite sequence of space–time cells, such that the infection spreads from one cell to the next.

In order to obtain this result, we will need to consider overlapping space–time cells. Let $\eta \geq 1$ be an integer which will represent the amount of overlap between cells. For each cube $i = (i_1, \dots, i_d)$ and time interval τ, define the *super cube* i as $\prod_{j=1}^{d}[(i_j - \eta)\ell, (i_j + \eta + 1)\ell]$ and the *super interval* τ as $[\tau\beta, (\tau + \eta)\beta]$. We define the *super cell* (i, τ) as the Cartesian product of the super cube i and the super interval τ.

In the following we will say a particle has displacement inside X' during a time interval $[t_0, t_0 + t_1]$, if the location of the particle at all times during $[t_0, t_0 + t_1]$ is inside $x + X'$, where x is the location of the particle at time t_0. We define a particle system on $\mathbb{Z}^d$ as a countable family of not necessarily unique elements of $\mathbb{Z}^d$, indexed by some countable set I, representing the locations of the particles belonging to the particle system. Let $(\Pi_s)_{s\geq 0}$ be a sequence of particles system on $\mathbb{Z}^d$, with Π_s representing the locations of the particles at time s. We say a particle system Π_s is distributed according to a Poisson random measure of intensity ζ, if for every $B \subset \mathbb{Z}^d$, $N(B)$ is a Poisson random variable with intensity $\zeta(B)$, where $N(B)$ is the number of particles belonging to Π_s that lie in B. We say an event E is *increasing* for $(\Pi_s)_{s\geq 0}$ if the fact that E holds for $(\Pi_s)_{s\geq 0}$ implies that it holds for all $(\Pi'_s)_{s\geq 0}$ for which $\Pi'_s \supseteq \Pi_s$ for all $s \geq 0$. We say an event E is *restricted* to a region $X \subset \mathbb{Z}^d$ and a time interval $[t_0, t_1]$ if it is measurable with respect to the σ-field generated by all the particles that are inside X at time t_0 and their positions from time t_0 to t_1. For an increasing event E that is restricted to a region X and time interval $[t_0, t_1]$, we have the following definition.

Definition 2. ν_E is called the *probability associated* to an increasing event E that is restricted to X and a time interval $[t_0, t_0 + t_1]$ if, for an intensity measure ζ, $\nu_E(\zeta, X, X', t_1)$ is the probability that E happens given that, at time t_0, the particles in X are a particle system distributed according to the Poisson random measure of intensity ζ and their motions from t_0 to $t_0 + t_1$ are independent continuous time random walks on the weighted graph (G, μ), where the particles are conditioned to have displacement inside X'.

For each $(i, \tau) \in \mathbb{Z}^{d+1}$, let $E_{\mathrm{st}}(i, \tau)$ be an increasing event restricted to the super cube i and the super interval τ. Here the subscript st refers to space–time. We say that a cell (i, τ) is *bad* if $E_{\mathrm{st}}(i, \tau)$ does not hold and *good* otherwise.

We will need a different way to index space–time cells, which we refer to as the *base-height index*. In the base-height index, we pick one of the d spatial dimensions and denote it as *height*, using index $h \in \mathbb{Z}$, while the remaining d space–time dimensions form the base, which we index by $b \in \mathbb{Z}^d$. In this way, for each space–time cell (i, τ) there will be $(b, h) \in \mathbb{Z}^{d+1}$ such that the base-height cell (b, h) corresponds to the space–time cell (i, τ). In other words, the base-height index is a (fixed) permutation of the coordinates of the space–time index which emphasizes one of the coordinates (either spatial or time) by making it the last coordinate. We will use the base-height index to define a $d+1$ dimensional object called the two-sided Lipschitz surface, for which one of the coordinates plays a special role—we will use the coordinate h of the base-height index for this purpose.

We analogously define the *base-height super cell* (b, h) to be the space–time super cell (i, τ), for which the base-height cell (b, h) corresponds to the space–time cell (i, τ). Similarly, we define $E_{\text{bh}}(b, h)$, the increasing event restricted to the super cell (b, h) that is the same as the event $E_{\text{st}}(i, \tau)$ for the space–time super cell (i, τ) that corresponds to the base-height super cell (b, h). Here, the subscript bh refers to the base-height index.

In order to prove Theorems 1 and 2, we will need a theorem from [7], which gives the existence of a two-sided Lipschitz surface F.

Definition 3. A function $F : \mathbb{Z}^d \to \mathbb{Z}$ is called a *Lipschitz function* if $|F(x) - F(y)| \leq 1$ whenever $\|x - y\|_1 = 1$.

Definition 4. A *two-sided Lipschitz surface* F is a set of base-height cells $(b, h) \in \mathbb{Z}^{d+1}$ such that for all $b \in \mathbb{Z}^d$ there are exactly two (possibly equal) integer values $F_+(b) \geq 0$ and $F_-(b) \leq 0$ for which $(b, F_+(b)), (b, F_-(b)) \in F$ and, moreover, F_+ and F_- are Lipschitz functions.

We say a space–time cell (i, τ) belongs to F if there exists a base-height cell $(b, h) \in F$ that corresponds to (i, τ). We say a two-sided Lipschitz surface F exists, if for all $b \in \mathbb{Z}^d$, we have $F_+(b) < \infty$ and $F_-(b) > -\infty$. For a positive integer D, we say a two-sided Lipschitz surface *surrounds* a cell (b', h') at distance D if any path $(b', h') = (b_0, h_0), (b_1, h_1), \ldots, (b_n, h_n)$ for which $\|(b_i, h_i) - (b_{i-1}, h_{i-1})\|_1 = 1$ for all $i \in \{1, \ldots n\}$ and $\|(b_n, h_n) - (b_0, h_0)\|_1 > D$, intersects with F.

We now present the main result from our paper [7], which holds for graphs where a local mixing result, such as the one in Theorem 5, hold. More precisely, for a graph satisfying Assumption 1 and (1) (which implies Assumption 2 holds) we have that Theorem 5 holds (with $S(x) = 1$ for all $x \in V$), which in turn gives that the following result from [7] holds. Recall that, for any $\rho \geq 2$, Q_ρ stands for the cube $[-\rho/2, \rho/2]^d$, and that λ is the intensity measure of the Poisson point process of particles as defined in Section 1.

Theorem 6. *Let $\mathcal{G} = ((\mathbb{Z}^d, E), \mu)$ with $d \geq 2$ be a nearest neighbor graph satisfying Assumption 1 and (1). There exist positive constants c_0, c_1 and c_2 such that the following holds. Tessellate G in space–time cells and super cells as described above for some $\ell, \beta, \eta > 0$ such that the ratio $\beta/\ell^2 < c_0$. Let $E_{\text{st}}(i, \tau)$ be an increasing event, restricted to the space–time super cell (i, τ). Fix $\epsilon \in (0, 1)$ and fix w such that*

$$w \geq \sqrt{\frac{\eta\beta}{c_2\ell^2} \log\left(\frac{8c_1}{\epsilon}\right)}.$$

Then, there exists a positive number α_0 that depends on ϵ, η and that ratio β/ℓ^2 so that if

$$\min\left\{C_M^{-1}\epsilon^2\lambda_0\ell^d, \log\left(\frac{1}{1 - \nu_{E_{\text{st}}}((1-\epsilon)\lambda, Q_{(2\eta+1)\ell}, Q_{w\ell}, \eta\beta)}\right)\right\} \geq \alpha_0, \tag{21}$$

a two-sided Lipschitz surface F where $E_{\mathrm{st}}(i,\tau)$ holds for all $(i,\tau) \in F$ exists almost surely. Furthermore, the surface surrounds the origin at a finite distance almost surely.

We now briefly explain the main conditions for the establishment of the above theorem. We will fix β/ℓ^2 to be an arbitrary, but small constant. The value of η defines the super cubes, which just model how much overlap we need between the cells of the tessellation (to allow information to propagate from one cell to its neighbors). Once these two parameters are fixed, we need to satisfy (21). First we need $C_M^{-1}\epsilon^2\lambda_0\ell^d \geq \alpha_0$. After fixing ϵ, this can be satisfied either by setting ℓ large enough (which makes the cells of the tessellation large), or by assuming that the density of particles λ_0 is large enough. Then we still need to make $\nu_{E_{\mathrm{st}}}((1-\epsilon)\lambda, Q_{(2\eta+1)\ell}, Q_{w\ell}, \beta) \geq 1-\exp(-\alpha_0)$. Usually E_{st} is a local event that becomes more and more likely by setting ℓ larger and larger; so having ℓ large enough suffices to satisfy this condition as well. The value of $\epsilon > 0$ is introduced so that in $\nu_{E_{\mathrm{st}}}$ we can consider a Poisson point process of particles of intensity measure $(1-\epsilon)\lambda$, slightly smaller than the actual intensity of particles. This slack is needed to restrict our attention to the particles that "behave well". Then the lower bound on w is to guarantee that, as particles move in $Q_{(2\eta+1)\ell}$ for time β, with high probability they do not leave $Q_{(2\eta+1)\ell+w\ell}$, allowing a better control of dependences between neighboring cells of the tessellation.

Recall that we want to show that the infection spreads with positive speed. Given a space–time tessellation of G and a local increasing event E_{st}, Theorem 6 gives the existence of a Lipschitz surface F on which E_{st} holds. Let $T = \ell^{5/3}$. We will define the increasing event $E_{\mathrm{st}}(i,\tau)$ to represent a single infected particle in the middle of the super cube i at time $\tau\beta$ infecting a large number of particles in that super cube by time $\tau\beta + T$, after which the infected particles move up to time $(\tau+1)\beta$, spreading to all of the cubes contained in the super cube.

Let (i,τ) be a space–time cell as defined previously. We consider that there is an infected particle in the center cube of the super cube i at time $\tau\beta$, that is, the particle is inside $\prod_{j=1}^d [i_j\ell, (i_j+1)\ell]$. Starting from time $\tau\beta$, we let the infected particle move and infect sufficiently many other particles by time $\tau\beta + T$. This is given in the lemma below.

Lemma 2. *Let τ, i and η be fixed and let $T = \ell^{5/3}$ as above. There exist positive constant C_1 such that the following holds for all large enough ℓ. Let $Q^* = \prod_{j=1}^d [(i_j-\eta)\ell, (i_j+\eta+1)\ell]$ and let $(\rho(t))_{\tau\beta\leq t\leq\tau\beta+T}$ be the path of an infected particle that starts in $\prod_{j=1}^d [i_j\ell, (i_j+1)\ell]$ and stays inside $\prod_{j=1}^d [(i_j-\eta+1)\ell, (i_j+\eta)\ell]$ during $[\tau\beta, \tau\beta+T]$. Assume that at time $\tau\beta$, the number of particles at each vertex $x \in Q^* \setminus \rho(\tau\beta)$ is a Poisson random variable of mean $\frac{\lambda_0}{2}\mu_x$. Let Υ be the set of these particles, and let $\Upsilon' \subset \Upsilon$ be the particles colliding with the path ρ, that is, for each particle of Υ' there exists a time $t \in [\tau\beta, \tau\beta+T]$ such that the particle is located at $\rho(t)$. Then, $|\Upsilon'|$ is a Poisson random variable of mean at least $C_1\lambda_0\ell^{1/3}$.*

Remark 4. We note that the statement of Lemma 2 is conditional on the path $(\rho(t))_{\tau\beta\leq t\leq\tau\beta+T}$. The bound we obtain is uniform across all such paths that we will consider later in Lemma 4, so we omit this in our notation.

Proof. For each time $t \in [\tau\beta, \tau\beta+T]$, let Ψ_t be the Poisson point process on V giving the locations at time t of the particles that belong to Υ. Since the particles that start in Q^* move around and can leave Q^*, we need to find a lower bound for the intensity of Ψ_t for times in $[\tau\beta, \tau\beta+T]$. Note that the infected particle we are tracking is not part of Ψ, since Ψ does not include particles located at $\rho(\tau\beta)$ at time $\tau\beta$.

We will need to apply heat kernel bounds from Proposition 1 to the particles in Q^*, so we need to ensure that the time intervals we consider are large enough for the proposition to hold. We will only consider times $t \in [\ell^{4/3}, T]$ so that for large enough ℓ, $t \geq \sup_{\substack{x \in Q^* \\ y \in Q^*}} \|x - y\|_1$ and so the heat kernel bounds from Proposition 1 hold. Then, we have that for all sites $x \in Q^*$ that are at least ℓ away from the boundary of Q^* and at any such time t the intensity of $\Psi_{\tau\beta+t}$ at vertex $x \in V$ is at least

$$\psi(x, \tau\beta + t) \geq \sum_{\substack{y \in Q^* \\ y \neq \rho(\tau\beta)}} \tfrac{\lambda_0}{2} \mu_y \cdot \mu_x q_t(y, x) = \tfrac{\lambda_0}{2} \mu_x \sum_{\substack{y \in Q^* \\ y \neq \rho(\tau\beta)}} \mathbb{P}_x[Y_t = y],$$

where we used in the last step that the heat kernel q_t is symmetric. We now use the exit time bound from Proposition 4 to get that

$$\sum_{y \in Q^*} \mathbb{P}_x[Y_t = y] \geq 1 - c_3 \exp\{-c_4 \ell^2 / t\}.$$

Next, we use that $\mathbb{P}_x[Y_t = y] = \mu_y q_t(x, y) \leq C_M q_t(x, y)$, and use Proposition 1 to account for the particles at $\rho(\tau\beta)$, yielding

$$\sum_{\substack{y \in Q^* \\ y \neq \rho(\tau\beta)}} \mathbb{P}_x[Y_t = y] \geq 1 - c_3 \exp\left\{-c_4 \ell^2 / t\right\} - C_M c_5 t^{-d/2}.$$

This gives that for any $t \in [\ell^{4/3}, T]$, the intensity of $\Psi_{\tau\beta+t}$ is at least

$$\psi(x, \tau\beta + t) \geq \tfrac{\lambda_0}{2} \mu_x (1 - c_3 \exp\{-c_4 \ell^2 / T\} - C_M c_5 \ell^{-2d/3}).$$

Let $[\tau\beta, \tau\beta + T]$ be divided into subintervals of length $W \in (0, T]$, where we set $W = \ell^{4/3}$ so that it is large enough to allow the use of the heat kernel bounds from Proposition 1. Let $J = \{1, \ldots, \lfloor T/W \rfloor\}$ and $t_j := \tau\beta + jW$. Then the intensity of particles that share a site with the initially infected particle at least once among times $\{t_1, t_2, \ldots, t_{\lfloor T/W \rfloor}\}$ is at least

$$\sum_{j \in J} \psi(\rho(t_j), t_j) \mathbb{P}_{\rho(t_j)}[Y_{r-t_j} \neq \rho(r) \; \forall r \in \{t_{j+1}, \ldots, t_{\lfloor T/W \rfloor}\}]$$

$$\geq \tfrac{\lambda_0}{2} C_M^{-1} (1 - c_3 \exp\{-c_4 \ell^2 / T\} - C_M c_5 \ell^{-2d/3}) \sum_{j \in J} \left(1 - \sum_{z > j} \mathbb{P}_{\rho(t_j)}[Y_{t_z - t_j} = \rho(t_z)] \right).$$

We want to make all of the terms of the sum over J positive, so we consider the term $\sum_{z>j} \mathbb{P}_{\rho(t_j)}[X_{t_z - t_j} = \rho(t_z)]$ and show that it is smaller than $\frac{1}{2}$ for large enough ℓ. To do this, we use that $\mathbb{P}_x[Y_t = y] = \mu_y q_t(x, y)$ with the heat kernel bounds from Proposition 1, which hold when $W \geq \ell^{4/3}$ and ℓ is large enough, to bound it from above by

$$\sum_{z > j} \mathbb{P}_{\rho(t_j)}[Y_{t_z - t_j} = \rho(t_z)] \leq \sum_{z > j} C_M C_{HK} (t_z - t_j)^{-d/2}$$

$$\leq C_M C_{HK} W^{-d/2} \sum_{z=1}^{T/W - j} z^{-d/2} \tag{22}$$

where C_{HK} is the constant coming from Proposition 1. Then, (22) can be bounded from above by

$$C_M C_{HK} W^{-d/2} \left(2 + \sum_{z=3}^{T/W - j} z^{-d/2} \right) \leq C_M C_{HK} W^{-d/2} \left(2 + \int_2^{T/W} z^{-d/2} dz \right). \tag{23}$$

Let C be a constant that can depend on C_{HK}, C_M and d. Then for $d = 2$, (23) it is smaller than $CW^{-1}\log(T/W)$, and for $d \geq 3$ the expression in (23) is smaller than $CW^{-d/2}$. Thus, setting ℓ large enough, both terms are smaller than $\frac{1}{2}$.

Then, as a sum of Poisson random variables, we get that Υ' is a Poisson random variable with a mean at least

$$\tfrac{\lambda_0}{2} C_M^{-1}(1 - c_3 \exp\{-2c_4\ell^2/T\} - C_M c_5 \ell^{-2d/3})\tfrac{T}{2W}.$$

Using that $T = \ell^{5/3}$ and setting ℓ large enough establishes the lemma, with C_1 being any constant satisfying $C_1 < \frac{C_M^{-1}}{4}$. □

Next we show that the particles from Lemma 2 move to nearby cells, spreading the infection.

Lemma 3. *Let $z = (z_1, \dots, z_d)$ with $z_j \in \{-\eta, -\eta + 1, \dots, \eta\}$ for all $j \in \{1, \dots d\}$, and fix the ratio β/ℓ^2. Let $A(i, \tau, N, z)$ be the event that given a set of $N > 0$ particles in $\prod_{j=1}^d [(i_j - \eta)\ell, (i_j + \eta + 1)\ell]$ at time $\tau\beta + T$, at least one of them is in $\prod_{j=1}^d [(i_j + z_j)\ell, (i_j + z_j + 1)\ell]$ at time $(\tau + 1)\beta$. Then, if ℓ is sufficiently large while keeping β/ℓ^2 fixed, we obtain*

$$\mathbb{P}[A(i, \tau, N, z)] \geq 1 - \exp\{-Nc_p\},$$

where c_p is a positive constant that is bounded away from 0 and depends only on d, η and the ratio β/ℓ^2.

Proof. Let $Q^* = \prod_{j=1}^d [(i_j - \eta)\ell, (i_j + \eta + 1)\ell]$ and $Q^{**} = \prod_{j=1}^d [(i_j + z_j)\ell, (i_j + z_j + 1)\ell]$. For $t^{2/3} \geq \sup_{\substack{x\in Q^* \\ y \in Q^{**}}} \|x - y\|_1$, define $p_t := \inf_{x\in Q^*} \sum_{y\in Q^{**}} \mathbb{P}_x[Y_t = y]$. Then, if we define $\mathrm{bin}(N, p_t)$ to be a binomial random variable of parameters $N \in \mathbb{N}$ and $p_t \in [0, 1]$, it directly follows that

$$\mathbb{P}[A(i, \tau, N, z)] \geq \mathbb{P}[\mathrm{bin}(N, p_t) \geq 1] \geq 1 - \exp\{-Np_t\}.$$

It remains to show that for $t = \beta - T$, we have that $p_t \geq c_p > 0$ for some constant c_p. We will use the heat kernel bounds for the pair x, y, which hold if $\|x - y\|_1^{3/2} \leq \beta - T$ for all $x \in Q^*$, $y \in Q^{**}$. Given the ratio β/ℓ^2, d and η, this is satisfied if ℓ is large enough. Then we have that

$$\begin{aligned} p_{\beta-T} &= \inf_{x\in Q^*} \sum_{y\in Q^{**}} \mathbb{P}_x[Y_{\beta-T} = y] \\ &\geq \inf_{x\in Q^*} C_M^{-1} \sum_{y\in Q^{**}} q_{\beta-T}(x, y) \\ &\geq \inf_{x\in Q^*} C_M^{-1} \sum_{y\in Q^{**}} c_1\beta^{-d/2} \exp\left\{-c_2 \frac{\|x-y\|_1^2}{\beta - T}\right\}. \end{aligned}$$

Now we use that x and y can be at most $c_\eta\ell$ apart where c_η is a constant depending on d and η only, and that $\beta - T \geq \beta/2$ for ℓ large enough. Hence,

$$\begin{aligned} p_{\beta-T} &\geq \inf_{x\in Q^*} C_M^{-1} \sum_{y\in Q^{**}} c_1\beta^{-d/2} \exp\left\{-c_2 \frac{2(c_\eta\ell)^2}{\beta}\right\} \\ &= C_M^{-1} c_1 \ell^d \left(\frac{1}{\beta}\right)^{d/2} \exp\left\{-c_2 \frac{2(c_\eta\ell)^2}{\beta}\right\} \\ &\geq c_p. \quad \square \end{aligned}$$

In the next lemma, we will tie together the results from Lemmas 2 and 3. In order to precisely describe the behavior of the particles involved, we say a particle x *collides* with particle y during a time interval $[t_0, t_1]$, if for at least one $t \in [t_0, t_1]$, x and y are at the same site.

Lemma 4. *Consider the super cell (i, τ). Assume that at each site $x \in \prod_{j=1}^{d}[(i_j - \eta)\ell, (i_j + \eta + 1)\ell]$ the number of particles at x at time $\tau\beta$ is an independent Poisson random variable of intensity $\frac{\lambda_0}{2}\mu_x$, and let Υ be the collection of such particles. Assume that, at time $\tau\beta$, there is at least one infected particle x_0 inside $\prod_{j=1}^{d}[i_j\ell, (i_j + 1)\ell]$. Let $E_{\mathrm{st}}(i, \tau)$ be the event that at time $(\tau + 1)\beta$, for all $i' \in \mathbb{Z}^d$ with $\|i - i'\|_\infty \le \eta$, there is at least one particle from Υ in $\prod_{j=1}^{d}[(i'_j)\ell, (i'_j + 1)\ell]$ that collided with x_0 during $[\tau\beta, \tau\beta + T]$. If ℓ is sufficiently large for* Lemmas 2 and 3 *to hold, then there exists a positive constant C such that*

$$\mathbb{P}[E_{\mathrm{st}}(i, \tau)] \ge 1 - \exp\{-C\lambda_0\ell^{1/3}\}.$$

Proof. We note that, by definition, the event $E_{\mathrm{st}}(i, \tau)$ is restricted to the super cube $\prod_{j=1}^{d}[(i_j - \eta)\ell, (i_j + \eta + 1)\ell]$ and time interval $[\tau\beta, (\tau + 1)\beta]$. We define the following 3 events.

- F_1: The initial infected particle x_0 never leaves $\prod_{j=1}^{d}[(i_j - \eta + 1)\ell, (i_j + \eta - 1)\ell]$ during $[\tau\beta, \tau\beta + T]$.
- F_2: Let C_1 be the constant from Lemma 2. During the time interval $[\tau\beta, \tau\beta + T]$ the initial infected particle x_0 collides with at least $\frac{C_1\lambda_0\ell^{1/3}}{2}$ different particles from Υ that are in the super cube $Q^{**} = \prod_{j=1}^{d}[(i_j - \eta)\ell, (i_j + \eta + 1)\ell]$ at time $\tau\beta + T$.
- F_3: Out of the $\frac{C_1\lambda_0\ell^{1/3}}{2}$ or more particles from F_2, at least one of them is in the cube $\prod_{j=1}^{d}[(i_j + k_j)\ell, (i_j + k_j + 1)\ell]$ at time $(\tau + 1)\beta$, for all $k = (k_1, \dots, k_d)$ for which $\prod_{j=1}^{d}[(i_j + k_j)\ell, (i_j + k_j + 1)\ell] \subset Q^{**}$.

By definition of the events, we clearly have that $\mathbb{P}[E_{\mathrm{st}}(i, \tau)] \ge \mathbb{P}[F_1 \cap F_2 \cap F_3]$.

Using Proposition 4 we have

$$\mathbb{P}[F_1] \ge 1 - C_2 \exp\{-C_3\ell^2/T\} = 1 - C_2 \exp\{-C_3\ell^{1/3}\} \tag{24}$$

for some positive constants C_2 and C_3. We observe that F_1 is restricted to the super cube $\prod_{j=1}^{d}[(i_j - \eta)\ell, (i_j + \eta + 1)\ell]$ and the time interval $[\tau\beta, \tau\beta + T]$.

For the event F_2, we apply Lemma 2 to get that the intensity of the Poisson point process of particles that are in Q^{**} at time $\tau\beta$ and collide with x_0 during $[\tau\beta, \tau\beta + T]$ is at least $\lambda_0 C_1\ell^{1/3}$ for some positive constant C_1. Since every particle that collides with x_0 enters $\prod_{j=1}^{d}[(i_j - \eta + 1)\ell, (i_j + \eta)\ell]$ during $[\tau\beta, \tau\beta + T]$, we can use Proposition 4 to bound the probability that the particle is inside of Q^{**} at time $\tau\beta + T$ by

$$1 - C_a \exp\left\{-\frac{C_b\ell^2}{T}\right\} = 1 - C_a \exp\{-C_b\ell^{1/3}\},$$

for some positive constants C_a and C_b. This term can be made as close to 1 as possible by having ℓ sufficiently large. We assume ℓ is large enough so that this term is larger than $2/3$. This gives that the intensity of the process of particles from Υ that collided with x_0 during $[\tau\beta, \tau\beta + T]$ and are in Q^{**} at time $\tau\beta + T$ is at least

$$\frac{2\lambda_0 C_1\ell^{1/3}}{3}.$$

Using Chernoff's bound (see Lemma 5) we have that

$$\mathbb{P}[F_2] \geq 1 - \exp\{-(2/3)^2 C_1 \lambda_0 \ell^{1/3}\}. \tag{25}$$

Note that, by construction, F_2 is restricted to the super cube Q^{**} and the time interval $[\tau\beta, \tau\beta + T]$. Furthermore, F_2 is clearly an increasing event.

We now turn to F_3. Using Lemma 3, and a uniform bound across the number of cubes inside a super cube, we have that

$$\mathbb{P}[F_3] \geq 1 - (2\eta + 1)^d \exp\{-\frac{C_1 \lambda_0 \ell^{1/3}}{2} c_p\}, \tag{26}$$

where c_p is a small but positive constant. Again, the event is restricted to the super cube Q^{**} and the time interval $[\tau\beta + T, (\tau + 1)\beta]$ and is an increasing event. Taking the product of the probability bounds in (24)–(26), we see that the probability that $E_{\mathrm{st}}(i, \tau)$ holds is at least

$$1 - \exp\{-C\lambda_0 \ell^{1/3}\}$$

for some constant C and all large enough ℓ. □

Proof of Theorem 1. We start by using Theorem 6. Set $\eta \in \mathbb{N}$ such that $\eta \geq d$ and set $\epsilon = 1/2$. Fix the ratio β/ℓ^2 small enough so that the lower bound for w is at most $2\eta + 1$, and then set $w = 2\eta + 1$. Assume ℓ is large enough so that Lemma 4 holds.

For each $(i, \tau) \in \mathbb{Z}^{d+1}$, define $E_{\mathrm{st}}(i, \tau)$ as in Lemma 4. This event is increasing in the number of particles, is restricted to the super cube i and time interval $[\tau\beta, (\tau + 1)\beta]$, and satisfies

$$\mathbb{P}[E_{\mathrm{st}}(i, \tau)] \geq 1 - \exp\{-C\lambda_0 \ell^{1/3}\},$$

for some constant C. Hence, letting $\lambda/2$ stand for the measure $\frac{\lambda}{2}(x) = \frac{\lambda_0 \mu_x}{2}$, we have

$$\log\left(\frac{1}{1 - \nu_{E_{\mathrm{st}}}(\frac{\lambda}{2}, Q_{(2\eta+1)\ell}, Q_{(2\eta+1)\ell}, \eta\beta)}\right) \geq C\lambda_0 \ell^{1/3},$$

which increases with ℓ, as does the term $\epsilon^2 \lambda_0 \ell^d$ in the condition of Theorem 6. Thus, setting ℓ large enough, we apply Theorem 6 which gives the existence of a two-sided Lipschitz surface F, on which the event $E_{\mathrm{st}}(i, \tau)$ holds. We also get that the surface is almost surely finite and that it surrounds the origin.

We now proceed to argue that the existence of the surface F implies that the infection spreads with positive speed. Since the two-sided Lipschitz surface F is finite and surrounds the origin, we have that in almost surely finite time, an infected particle started from the origin will enter some cube $\prod_{j=1}^{d} [i_j \ell, (i_j + 1)\ell]$ for which (i, τ) is in F. We call this the *central cube* of (i, τ). Once that holds, the starting assumption of $E_{\mathrm{st}}(i, \tau)$ from Lemma 4 is satisfied for the super cell (i, τ), and the event $E_{\mathrm{st}}(i, \tau)$ holds. By the definition of $E_{\mathrm{st}}(i, \tau)$ this means that the initial infected particle for the super cell (i, τ) infects a large number of other particles, which spread the infection to the central cube of $(i', \tau + 1)$ for all $i' \in \mathbb{Z}^d$ such that $\|i' - i\|_\infty \leq \eta$.

Let (b, h) be the base-height index of the cell $(i, \tau) \in F$. Recall that h is one of the spatial dimensions. We will also select one of the $d - 1$ spatial dimensions from b and denote it b_1. Let $b' \in \mathbb{Z}^d$ be obtained from b by increasing the time dimension from τ to $\tau + 1$, and by increasing the chosen spatial dimension from b_1 to $b_1 + 1$. Since $\|b - b'\|_1 = 2$, we can choose $h' \in \mathbb{Z}$ such that $(b', h') \in F$ and $|h - h'| \leq 2$, where the latter holds by the Lipschitz property of F. Therefore, there must exist $i' \in \mathbb{Z}^d$ such that $(i', \tau + 1)$ is the space–time super cell corresponding to (b', h') and $\|i - i'\|_\infty \leq 1$. Hence, at time $(\tau + 1)\beta$, there is an infected particle in the central cube of the super cell i'.

We can then recursively repeat this procedure for the super cell $(i', \tau + 1)$, since $E_{\text{st}}(i', \tau + 1)$ holds. Repeating this process we obtain that the infection spreads by a distance of at least ℓ in time β in the chosen spatial direction. Consequently

$$\liminf_{t\to\infty} \frac{\|I_t\|_1}{t} > 0 \quad \text{almost surely.} \qquad \square$$

In order to prove Theorem 2, we can follow the same steps as in the proof of Theorem 1 with the additional consideration that we have to ensure that the relevant infected particles do not recover too quickly. For that, we will require that all the particles involved do not recover for at least β.

Proof Theorem 2. Recall the definition of Υ and ρ from Lemma 2 and of $E_{\text{st}}(i, \tau)$ from Lemma 4. Let $E'_{\text{st}}(i, \tau)$ be the event that $E_{\text{st}}(i, \tau)$ holds, and that the particles in Υ and the initial infected particle whose path is ρ do not recover during $[\tau\beta, (\tau + 1)\beta]$. Since each such particle does not recover during $[\tau\beta, (\tau + 1)\beta]$ with probability $\exp\{-\gamma\beta\}$, for Lemma 2 we consider that for each $x \in Q^* \setminus \rho(\tau\beta)$ the number of particles at x at time $\tau\beta$ that do not recover during $[\tau\beta, (\tau + 1)\beta]$ is a Poisson random variable of intensity $\frac{\lambda_0}{2}\mu_x \exp\{-\lambda\beta\}$. Thus, once η, β and ℓ are fixed, setting γ small enough gives that $E'_{\text{st}}(i, \tau)$ holds with probability at least

$$1 - (1 - \exp\{-\gamma\beta\}) - \exp\{-C\lambda_0 \exp\{-\gamma\beta\}\ell^{1/3}\}$$

for some positive constant C, where the term inside the parenthesis accounts of the probability that the initial infected particles recover during $[\tau\beta, (\tau + 1)\beta]$. We now follow the same steps as in the proof of Theorem 1 to get that the two-sided Lipschitz surface F on which the increasing event $E'_{\text{st}}(i, \tau)$ holds exists, is finite and surrounds the origin almost surely. This gives that an initially infected particle that is at the origin at time 0 has a strictly positive probability of surviving long enough to enter a cell of the two-sided Lipschitz surface. Once on the surface, the infection survives indefinitely by the definition of $E'_{\text{st}}(i, \tau)$. Hence

$$\mathbb{P}\left[\|I_t\|_1 \geq c_1 t \text{ for all } t \geq c_3\right] \geq c_2. \qquad \square$$

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Appendix. Standard large deviation results

Lemma 5 (*Chernoff Bound for Poisson*). *Let P be a Poisson random variable with mean λ. Then, for any $0 < \epsilon < 1$,*

$$\mathbb{P}[P < (1 - \epsilon)\lambda] < \exp\{-\lambda\epsilon^2/2\}$$

and

$$\mathbb{P}[P > (1 + \epsilon)\lambda] < \exp\{-\lambda\epsilon^2/4\}.$$

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