



# Transience Versus Recurrence for Scale-Free Spatial Networks

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**Abstract.** Weight-dependent random connection graphs are a class of local network models that combine scale-free degree distribution, small-world properties and clustering. In this paper we discuss recurrence or transience of these graphs, features that are relevant for the performance of search and information diffusion algorithms on the network.

**Keywords:** Random-connection model · Recurrence · Transience · Scale-free percolation · Preferential attachment · Boolean model

## 1 Introduction and Statement of Results

### 1.1 Motivation

In the age of “big data” we are increasingly faced with data that is not linearly structured and instead organised in the form of networks. Algorithmic processing of such data is often dependent on the topological connectivity properties of the network. In this paper we therefore investigate finer connectivity features of a range of random network models. Features shared by these models are:

- They are *scale-free*, i.e., the proportion of nodes with  $k$  neighbours is of order  $k^{-\tau+o(1)}$  for some *power law exponent*  $\tau$ .
- They are *clustering*, i.e., local neighbourhoods of a node have a much higher connectivity than the overall network.
- They are *small worlds*, i.e., the graph distances are no more than polylogarithmic with respect to the system size.

Under the further assumption that the power law exponent  $\tau$  is sufficiently small, the models have the following additional features:

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B. Kamiński et al. (Eds.): WAW 2020, LNCS 12091, pp. 96–110, 2020.

[https://doi.org/10.1007/978-3-030-48478-1\\_7](https://doi.org/10.1007/978-3-030-48478-1_7)

- They are *ultrasmall*, i.e., the shortest path between two typical nodes in the graph is doubly logarithmic in the size of the graph.
- They are *robust*, i.e., if an arbitrarily large proportion of links is randomly removed the qualitative features of the network remain unchanged.

A prototype of such a network is the *age-based spatial preferential attachment model* introduced in [8]. In this model nodes arrive after exponential waiting times and upon birth are placed randomly on the unit torus  $\mathbb{T}^d$ . They connect independently to all existing nodes with a probability which is a decreasing function of the spatial distance and the birth times of both vertices. This network model is a simplification of the spatial preferential attachment model in [16] and, in a less general setup, in [1], which however is believed to retain all essential features of the more complex original spatial preferential attachment model.

The major tool to study the age-based spatial preferential attachment model is to look at a local limit graph on  $\mathbb{R}^d$ . Such a graph describes the scaled neighbourhoods of a typical vertex in the network at a large time; long term averaged features of the network are reflected in the features of the limiting graph. It is shown in [8] that the limiting graph for the age-based spatial preferential attachment model is the *age-dependent random connection model*, which is a special case of the class of weight-dependent random connection models studied in this paper and introduced below. This connection is used in [8] to show scale-freeness and clustering, and in [9, 11] to identify regimes of robustness and ultrasmallness of the age-based spatial preferential attachment model.

In the context of information propagation, further properties of networks are relevant: How long does it take for the information to propagate and reach a set of targets for the first time? Does a target node receive the information at all (or is it possible that information bypasses it)? Can a single source result in information reaching a target in more than one way? Conversely, can information travel through the network without being “detected” by a large proportion of the network? Such questions are tightly connected to (and often well described by) the behaviour of a random walker on the network. In many cases, the behaviour of the walker is crucial for the development of random walk based search strategies, cf. [27]. The present paper addresses the problem of recurrence or transience of the limiting graph. Recurrence means that a random walker returns infinitely often to its origin and it is a prerequisite for the functioning of many search and information diffusion algorithms on networks [23].

## 1.2 The Weight-Dependent Random Connection Model

We study the transient/recurrent behaviour of a class of infinite graphs that, although not necessarily built as a limit of a growing sequence of finite graphs as in our motivating example, are built using similarly simple rules to connect pairs of vertices. We call this class of graphs the *weight-dependent random connection model* and we now describe the building principles they all have in common.

The vertex set of the model is a Poisson process of unit intensity on  $\mathbb{R}^d \times [0, 1]$ . We think of a Poisson point  $\mathbf{x} = (x, s)$  as a *vertex at position  $x$  with weight  $s^{-1}$* .

Two vertices  $\mathbf{x} = (x, s)$  and  $\mathbf{y} = (y, t)$  are connected by an edge with probability  $\varphi(\mathbf{x}, \mathbf{y})$  for a connectivity function

$$\varphi: (\mathbb{R}^d \times [0, 1]) \times (\mathbb{R}^d \times [0, 1]) \rightarrow [0, 1], \quad (1)$$

satisfying  $\varphi(\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{y}, \mathbf{x})$ . Connections between different (unordered) pairs of vertices occur independently. We assume throughout that  $\varphi$  has the form

$$\varphi(\mathbf{x}, \mathbf{y}) = \varphi((x, s), (y, t)) = \rho(\beta^{-1} g(s, t) |x - y|^d) \quad (2)$$

for a non-increasing, integrable *profile function*  $\rho: \mathbb{R}_+ \rightarrow [0, 1]$ , a *percolation parameter*  $\beta > 0$  and a *kernel function*  $g: [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+$ . The parameter  $\beta$  controls the edge density in a monotone way; increasing  $\beta$  increases the number of edges connected to a vertex at the origin. Varying  $\beta$  can also be interpreted as rescaling distances, and therefore it is equivalent to varying the intensity of the underlying Poisson process. We assume further that  $g$  is non-increasing in both arguments; and  $\rho$  is non-increasing, so that vertices whose positions are far apart are less likely to be connected. Without loss of generality we scale the profile function as

$$\int_{\mathbb{R}^d} \rho(|x|^d) dx = 1. \quad (3)$$

Then it is easy to see that the degree distribution of a vertex does not depend on  $\rho$  (see for example [8, Proposition 4.1]); it does however influence the likelihood of long edges.

We next give concrete examples for the kernel function  $g$ , and demonstrate that our setup yields a number of well-known models in continuum percolation theory. We define the functions in terms of a parameter  $\gamma \in (0, 1)$ , which determines the strength of the influence of the vertex weight on the connection probabilities; large  $\gamma$  correspond to strong favouring of vertices with large weight. The models considered below are all scale-free with power-law exponent

$$\tau = 1 + \frac{1}{\gamma},$$

which means that, in probability as  $N \rightarrow \infty$ ,

$$\frac{\# \text{ vertices } x \in [-N, N]^d \text{ with degree } k}{\# \text{ vertices } x \in [-N, N]^d} \rightarrow \mu(k), \text{ and } \mu(k) = k^{-\tau+o(1)}. \quad (4)$$

(A) We define the *plain kernel* as

$$g^{\text{plain}}(s, t) = 1. \quad (5)$$

In this case we have no dependence on the weights. If  $\rho(r) = 1_{[0, a]}$  for  $a = d/\omega_d$  and  $\omega_d$  is the area of the unit sphere in  $\mathbb{R}^d$ , this gives the *Gilbert disc model* with radius  $\sqrt[d]{\beta a}$ . Functions  $\rho$  of more general form lead to the (ordinary) *random connection model*, including in particular a continuum version of *long-range percolation* when  $\rho$  has polynomial decay at infinity.

(B) We define the *sum kernel* as

$$g^{\text{sum}}(s, t) = \left( s^{-\gamma/d} + t^{-\gamma/d} \right)^{-d}. \quad (6)$$

Interpreting  $(as^{-\gamma})^{1/d}$ ,  $(at^{-\gamma})^{1/d}$  as random radii and letting  $\rho(r) = 1_{[0,a]}$  we get the *Boolean model* in which two vertices are connected by an edge if the associated balls intersect. We get a further variant of the model with the *min-kernel* defined as

$$g^{\text{min}}(s, t) = (s \wedge t)^\gamma,$$

which, as  $g^{\text{sum}} \leq g^{\text{min}} \leq 2g^{\text{sum}}$ , shows qualitatively the same behaviour.

(C) For the *max-kernel* defined as

$$g^{\text{max}}(s, t) = (s \vee t)^{1+\gamma},$$

we may choose any  $\gamma > 0$ . This is a continuum version and generalization of the ultra-small scale-free geometric networks of Yukich [28], which is also parametrized to have power-law exponent  $\tau = 1 + \frac{1}{\gamma}$ .

(D) A particularly interesting case is the *product kernel*

$$g^{\text{prod}}(s, t) = s^\gamma t^\gamma, \quad (7)$$

which leads to a continuum version of the *scale-free percolation* model of Deijfen et al. [4, 13], see also [5, 6]. This model combines features of scale-free random graphs and polynomial-decay long-range percolation models (for suitable choice of  $\rho$ ).

(E) Our final example for  $g$  is the *preferential attachment kernel*

$$g^{\text{pa}}(s, t) = (s \vee t)^{1-\gamma} (s \wedge t)^\gamma, \quad (8)$$

which gives rise to the *age-dependent random connection model* introduced by Gracar et al. [8] as local weak limit of the age-based preferential attachment model, which is a simplification and approximation of the spatial preferential attachment model in Jacob and Mörters [16]. In this model,  $s$  and  $t$  actually play the role of the birth times of vertices in the underlying dynamic network. This model also combines scale-free degree distributions with power-law exponent  $\tau = 1 + \frac{1}{\gamma}$  and long edges in a natural way, but has a fundamentally different graph topology, as we will see.

The above listed kernels represent only some of the possible choices. We refer the reader to Table 1 for a short literature survey of the terminology under which the above kernels appear in the literature.

### 1.3 Main Results

We now focus on a profile function with polynomial decay

$$\lim_{v \rightarrow \infty} \rho(v) v^\delta = 1 \quad \text{for a parameter } \delta > 1, \quad (9)$$

**Table 1.** Terminology of the models in the literature.

Vertices	Profile	Kernel	Name and reference
Poisson	Indicator	Plain	Random geometric graph, Gilbert disc model [24]
Poisson	General	Plain	Random connection model [20]
			Soft random geometric graph [25]
Lattice	Polynomial	Plain	Long-range percolation [2]
Poisson	Indicator	Sum	Boolean model [12, 21]
Lattice	Indicator	Max	Ultra-small scale-free geometric networks [28]
Poisson	Indicator	Min	Scale-free Gilbert graph [15]
Lattice	Polynomial	Prod	Scale-free percolation [4, 13]
Poisson	Polynomial	Prod	Inhomogeneous long-range percolation [5]
			Continuum scale-free percolation [6]
Poisson	General	Prod	Geometric inhomogeneous random graphs [3]
Poisson	General	Pa	Age-dependent random connection model [8]

renormalised appropriately to satisfy (3), and fix one of the kernel functions described above. We keep  $\gamma, \delta$  fixed and study the resulting graph  $\mathcal{G}^\beta$  as a function of  $\beta$ . Note that our assumptions  $\delta > 1$  and  $\gamma < 1$  guarantee that  $\mathcal{G}^\beta$  is locally finite for all values of  $\beta$ , since it can quickly be checked that the degree of every vertex is Poisson distributed, cf. [8, p. 315 and Proposition 4.1]. We informally define  $\beta_c$  as the infimum over all values of  $\beta$  such that  $\mathcal{G}^\beta$  contains an infinite component (henceforth the infinite *cluster*). If  $d \geq 2$ , we always have  $\beta_c < \infty$ , cf. [13]. General arguments in [7] yield that there is at most one infinite component of  $\mathcal{G}^\beta$ , and hence there is a unique infinite component whenever  $\beta > \beta_c$ . We study the properties of this infinite cluster.

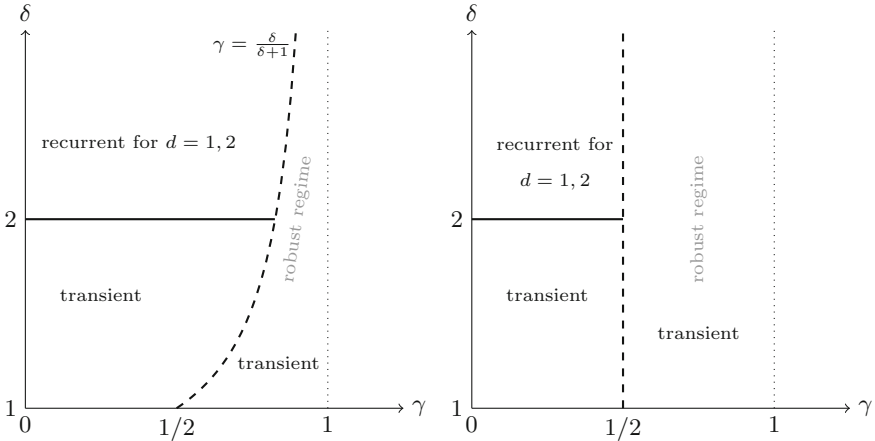
Two cases correspond to different network topologies.

- If  $\gamma > \frac{1}{2}$  for the product kernel,  $\gamma > 0$  for the max kernel, or  $\gamma > \frac{\delta}{\delta+1}$  for the preferential attachment, min or sum kernels we have  $\beta_c = 0$ , i.e. there exists an infinite cluster irrespective of the edge density, see [13] for product, [28] for max and [11, 17] for all other kernels. We say that this is the *robust case*.
- Otherwise, if  $\gamma < \frac{1}{2}$  for the product kernel, see [13], or if  $\gamma < \frac{\delta}{\delta+1}$  for the preferential attachment, min or sum kernels, we have  $\beta_c > 0$ . This was recently shown in [11]. In this case we say we are in the *non-robust case*.

Our main interest is whether the infinite cluster is recurrent (i.e., simple random walk on the cluster returns to its starting point almost surely), or transient (i.e., simple random walk on the cluster never returns to its starting point with positive probability). Our results are summarized in the following theorem.

**Theorem 1 (Recurrence vs. transience of the weight-dependent random connection model).** *Consider the weight-dependent random connection model with profile function satisfying (9).*

- (a) For preferential attachment kernel, sum kernel, or min kernel and  $\beta > \beta_c$ , the infinite component is
  - transient if either  $1 < \delta < 2$  or  $\gamma > \delta/(\delta + 1)$ ;
  - recurrent if  $d \in \{1, 2\}$ ,  $\delta > 2$  and if  $\gamma < \delta/(\delta + 1)$ .
- (b) For the product kernel and  $\beta > \beta_c$ , the infinite component is
  - transient if either  $1 < \delta < 2$  or  $\gamma > 1/2$ ;
  - recurrent if  $d \in \{1, 2\}$ ,  $\delta > 2$  and if  $\gamma < 1/2$ .
- (c) For the max kernel and  $\beta > \beta_c$ , the infinite component is transient.



**Fig. 1.** The different phases in Theorem 1: Left: preferential attachment kernel. Right: product kernel. The dashed line separates the robust from the non-robust phase.

For a summary of the results we refer to Fig. 1. We describe the proof of this theorem in Sects. 3 and 4, and refer to the journal version [10] for the full argument. We emphasise again that the characterisation of the regimes for this large class of graphs yields important information about them, useful for example in order to determine whether random walker algorithms will be able to properly scale to larger graphs or not.

**Remarks:**

- Loosely speaking, for the models in (a) and (b) the walk can travel to infinity using long edges if there are enough of them, i.e. if  $\delta < 2$ . For the models in (a) the walk can also use that vertices of ever increasing weight can be reached using a pool of intermediate vertices, which is big enough if  $\gamma > \delta/(\delta + 1)$ . For the model in (b) with  $\gamma > \frac{1}{2}$  and the model in (c) with  $0 < \gamma < 1$  the walk can travel directly between vertices of increasing weight without using intermediate edges.

- When  $\delta > 2$  and  $\gamma < \frac{\delta}{\delta+1}$  for the preferential attachment kernel resp.  $\gamma < \frac{1}{2}$  for the product kernel, we expect that the long-range and scale-free effects do not influence the behaviour of the random walk, so that for  $d \geq 3$  the infinite cluster would be transient. A mathematical proof of transience in this regime (even for long-range percolation) has not yet been found. We plan to address this in future work.

## 2 The Weight-Dependent Random Connection Model

**Construction As a Point Process on  $(\mathbb{R}^d \times [0, 1])^{[2]} \times [0, 1]$ .** We give now a more formal construction of our model. To this end, we extend the construction given in [14, Sections 2.1 and 2.2] by additional vertex marks (the *weight* or birth time). For further constructions, see Last and Ziesche [19] and Meester and Roy [22]. We construct the random connection model as a deterministic functional  $\mathcal{G}_\varphi(\xi)$  of a suitable point process  $\xi$ . Let  $\eta$  denote a unit intensity  $\mathbb{R}^d$ -valued Poisson point process, which we can write as

$$\eta = \{X_i : i \in \mathbb{N}\}; \tag{10}$$

such enumeration is possible, cf. [18, Corollary 6.5]. In order to define random walks on the random connection model, it is convenient to have a designated (starting) point, and we therefore add an extra point  $X_0 = 0$  and thereby get a Palm version of the Poisson point process.

We further equip any Poisson point  $X_i$  ( $i \geq 0$ ) with an independent mark  $S_i$  drawn uniformly from the interval  $(0, 1)$ . This defines a point process  $\eta' := \{\mathbf{X}_i = (X_i, S_i) : i \in \mathbb{N}_0\}$  on  $\mathbb{R}^d \times (0, 1)$ . Let  $(\mathbb{R}^d \times (0, 1))^{[2]}$  denote the space of all sets  $e \subset \mathbb{R}^d \times (0, 1)$  with exactly two elements; these are the potential edges of the graph. We further introduce independent random variables  $(U_{i,j} : i, j \in \mathbb{N}_0)$  uniformly distributed on the unit interval  $(0, 1)$  such that the double sequence  $(U_{i,j})$  is independent of  $\eta'$ . Using  $<$  for the strict lexicographical order on  $\mathbb{R}^d$ , we can now define

$$\xi := \{(\{(X_i, S_i), (X_j, S_j)\}, U_{i,j}) : X_i < X_j, i, j \in \mathbb{N}_0\}, \tag{11}$$

which is a point process on  $(\mathbb{R}^d \times (0, 1))^{[2]} \times (0, 1)$ . Mind that  $\eta'$  might be recovered from  $\xi$ . Even though the definition of  $\xi$  does depend on the ordering of the points of  $\eta$ , its distribution does not.

We can now define the weight-dependent random connection model  $\mathcal{G}_\varphi(\xi)$  as a deterministic functional of  $\xi$ ; its vertex and edge sets are given as

$$\begin{aligned} V(\mathcal{G}_\varphi(\xi)) &= \eta', \\ E(\mathcal{G}_\varphi(\xi)) &= \{\{\mathbf{X}_i, \mathbf{X}_j\} \in V(\mathcal{G}_\varphi(\xi))^{[2]} : X_i < X_j, U_{i,j} \leq \varphi(\mathbf{X}_i, \mathbf{X}_j), i, j \in \mathbb{N}_0\}. \end{aligned}$$

Only in this section we write  $\mathcal{G}_\varphi(\xi)$  in order to make the dependence on the connection function  $\varphi$  explicit; in the following sections we will fix a kernel function as well as the parameters  $\delta$  and  $\gamma$ , and therefore only write  $\mathcal{G}^\beta = \mathcal{G}^\beta(\xi)$ .

**Percolation.** Our construction ensures that  $\mathbf{0} := (X_0, S_0) \in V(\mathcal{G}_\varphi(\xi))$ . We now write  $\{0 \leftrightarrow \infty\}$  for the event that the random graph  $\mathcal{G}_\varphi(\xi)$  contains an infinite self-avoiding path  $(v_1, v_2, v_3, \dots)$  of vertices with  $v_i \in V(\mathcal{G}_\varphi(\xi))$  ( $i \in \mathbb{N}$ ) such that  $\{\mathbf{0}, v_1\}, \{v_1, v_2\}, \{v_2, v_3\} \dots \in E(\mathcal{G}_\varphi(\xi))$ , and we say that in this case the graph *percolates*. We denote the percolation probability by

$$\theta(\beta) = \mathbb{P}(0 \leftrightarrow \infty \text{ in } \mathcal{G}_\varphi(\xi));$$

for the probability that this happens; this quantity is well-defined by the monotonicity of the right-hand side in  $\beta$ . This allows us to define the critical percolation threshold as

$$\beta_c := \inf\{\beta > 0 : \theta(\beta) > 0\} \geq 0. \tag{12}$$

**Random Walks.** We recall that, as  $\gamma < 1$ , the resulting graph  $\mathcal{G}_\varphi(\xi)$  is locally finite, i.e.

$$\sum_{\mathbf{y} \in V(\mathcal{G}_\varphi(\xi))} \mathbb{1}\{\{\mathbf{x}, \mathbf{y}\} \in E(\mathcal{G}_\varphi(\xi))\} < \infty \quad \text{for all } \mathbf{x} \in V(\mathcal{G}_\varphi(\xi)) \text{ almost surely,} \tag{13}$$

cf. [8, p. 315 and Proposition 4.1]. Given  $\mathcal{G}_\varphi(\xi)$  with  $0 \leftrightarrow \infty$  we define the *simple random walk* on the random graph  $\mathcal{G}_\varphi(\xi)$  as the discrete-time stochastic process for which  $X_0 = 0$  and

$$P^{\mathcal{G}_\varphi(\xi)}(X_n = y \mid X_{n-1} = x) = \frac{\mathbb{1}\{\{\mathbf{x}, \mathbf{y}\} \in E(\mathcal{G}_\varphi(\xi))\}}{\sum_{\mathbf{z} \in V(\mathcal{G}_\varphi(\xi))} \mathbb{1}\{\{\mathbf{x}, \mathbf{z}\} \in E(\mathcal{G}_\varphi(\xi))\}}$$

for any  $\mathbf{x}, \mathbf{y} \in V(\mathcal{G}_\varphi(\xi))$  and  $n \in \mathbb{N}$ . We say that  $\mathcal{G}_\varphi(\xi)$  is *recurrent* if

$$P^{\mathcal{G}_\varphi(\xi)}(\exists n \geq 1 : X_n = 0) = 1,$$

otherwise we say that it is *transient*.

### 3 Transience

In this section we outline the key steps needed to prove the transience statement of Theorem 1. Throughout, we write  $\mathcal{G}^\beta$  instead of  $\mathcal{G}_\varphi(\xi)$  to stress that kernel and profile are fixed and the percolation parameter is  $\beta$ .

#### 3.1 Transience in the Robust Case

Proving transient behaviour for the robust case hinges on a renormalisation sequence argument. Heuristically, we consider a finite box in  $\mathbb{R}^d$  and the largest cluster of connected vertices inside of this box. Then, if the box is chosen sufficiently large, the probability that this cluster represents a positive proportion of the entire vertex set is increasing, as is the probability that within this cluster a vertex with weight greater than some predetermined value exists. When these



two conditions are satisfied, we consider this box good. We now, roughly speaking, repeat this argument for a considerably larger box. We break this large box into disjoint boxes of the previous scale and consider only those boxes which satisfy the two conditions (which occurs independently and with uniform probability for all boxes). Then, we call the bigger box good whenever a sufficiently large proportion of the boxes contained therein are good, they are sufficiently well connected with each other and there exists a vertex in this newly constructed cluster with weight greater than an even larger predetermined value.

Repeating this procedure at greater and greater scales we obtain a renormalised graph sequence that is contained in the infinite component of the graph and can be shown to be transient with a fairly straightforward argument. We formalise this statement in the following two results and leave the proof of Theorem 2 for the full version of this paper [10].

Given a graph  $G = (V, E)$  and a sequence  $\{C_n\}_{n=1}^\infty$  let  $V_l(j_l, \dots, j_1)$  with  $l \in \mathbb{N}_0$  and  $j_n \in \{1, \dots, C_n\}$  be an element of the vertex set  $V$ . Furthermore, let  $V_0$  be some arbitrary vertex. Now let for  $l \geq m > 1$

$$V_l(j_l, \dots, j_m) = \bigcup_{j_{m-1}=1}^{C_{m-1}} \dots \bigcup_{j_1=1}^{C_1} V_l(j_l, \dots, j_1).$$

We call the sets  $V_l(j_l, \dots, j_m)$  *bags*, and the numbers  $C_n$  *bag sizes*.

**Definition 1.** *We say that the graph  $G = (V, E)$  is renormalized for the sequence  $\{C_n\}_{n=1}^\infty$  if we can construct an infinite sequence of graphs such that the vertices of the  $l$ -th stage graph are labelled by  $V_l(j_l, \dots, j_1)$  for all  $j_n \in \{1, \dots, C_n\}$ , and such that for every  $l \geq m > 2$ , every  $j_l, \dots, j_{m+1}$ , and all pairs of distinct  $u_m, w_m \in \{1, \dots, C_m\}$  and  $u_{m-1}, w_{m-1} \in \{1, \dots, C_{m-1}\}$  there is an edge in  $G$  between a vertex in the bag  $V_l(j_l, \dots, j_{m+1}, u_m, u_{m-1})$  and a vertex in the bag  $V_l(j_l, \dots, j_{m+1}, w_m, w_{m-1})$ .*

The underlying intuition is that every  $n$ -th stage bag contains  $C_n (n - 1)$ -stage bags, each one of which contains again  $C_{n-1} (n - 2)$ -stage bags. Every pair of  $(n - 2)$ -stage bags in a  $n$  stage bag is connected by an edge between one of the vertices in the bags.

**Lemma 1 (Berger, [2, Lemma 2.7]).** *A graph renormalized for the sequence  $C_n$  is transient if  $\sum_{n=1}^\infty C_n^{-1} < \infty$ .*

**Theorem 2 (Product and max kernel contain a renormalised graph sequence).** *Let  $\beta > \beta_c$  for the chosen kernel. If  $\gamma > \frac{1}{2}$  for  $g^{prod}$  or  $\gamma > 0$  for  $g^{max}$ , then the infinite connected component contains a graph renormalized for the sequence  $C_n = (n + 1)^{2d}$  almost surely. Consequently, the infinite component is transient.*

Before we proceed to argue the result for the remaining kernels from Theorem 1, we first give a heuristic argument as to why the proof for the max and product kernel does not work for the rest. In order to demonstrate the argument, we assume for the moment that the profile function  $\rho$  is the indicator

function  $\mathbb{1}_{[0,c]}$  for some constant  $c$ . The first key observation is that two nodes of the graph that are far apart can only be connected if both of their respective weights are large. This is especially clear in the max kernel case, since a connection between two nodes at distance  $v$  is then only possible when *both* of their weights are greater than  $(v^d/(\beta c))^{1/(1+\gamma)}$ . A similar observation can be made for the product kernel - if one of the weights is small, the other weight must be considerably bigger to ensure their product is large enough.

For the other kernels, both weights being large is similarly beneficial; it should be remarked however that the probability of both weights being large is not sufficiently offset by the increase in the connection probability. Unlike in the above example, only the first heavy weight leads to a big increase in the probability of a connection existing (the second weight has a considerably smaller effect on the probability). In contrast to the max and product kernel, however, if the profile function has sufficiently heavy tails at infinity, an alternative strategy exists. We can connect pairs of nodes with large weights through a connector node with a comparatively small weight. Intuitively, since the smaller of the weights in a pair does not affect the connection probability of two nodes for the min kernel (and affects the probability at a lower order than the large weight in the sum and pa kernel), we can attempt to connect two far away nodes with large weights through nodes of smaller weight. Their large number then makes the probability of successfully finding such two-step connection sufficiently high to again obtain a renormalised graph sequence. It is helpful to interpret this event (i.e., the existence of a connector node through which both nodes of the pair are connected) as the existence of a *bridged connection* in a new graph with the same vertex set (Fig. 2). Note however that a connection in this new graph does not correspond to a specific tuple of a connector node and two corresponding edges; instead, a bridged connection corresponds to the *existence* of such a tuple. Then, one can roughly speaking use the same strategy as before in this new graph.

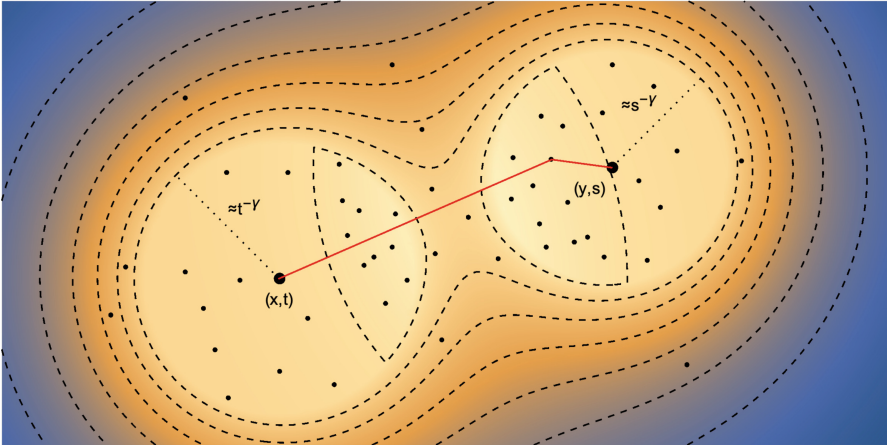
Therefore, although the construction of Theorem 2 does not yield the desired renormalised graph sequence for the remaining kernels when using direct connections, it does lead to the stated result when using the bridged connections instead. We state the main properties that hold for these connections in the following proposition.

**Proposition 1 (Occurrence of bridged connections).** *Let  $(x, t)$  and  $(y, s)$  be two nodes of the graph with  $t > s$ . Then there exists a positive constant  $C$  such that the probability that there exists a bridged connection between the two vertices is at least*

$$1 - \exp\{-Cs^{-\gamma}\rho(\beta^{-1}t^\gamma(s^{-\frac{\gamma}{d}} + |x - y|)^d)\}.$$

*Furthermore, this probability is monotonically decreasing in  $|x - y|$ .*

A direct consequence of Proposition 1 is that one can, using the same construction as used to prove Theorem 2, obtain a renormalised graph sequence satisfying the conditions of Lemma 1 which leads to the following result.



**Fig. 2.** The diagram illustrates the intensity of the Poisson point process of all points that are connected to  $(x, t)$  and  $(y, s)$  for the pa kernel and a large  $\delta$ . Intuitively, the area where such points are probable (contained roughly within the 4th innermost contour line) is sufficiently large to make the existence of a *bridged connection* (like the one in red) more probable than the direct connection (which would fall well outside the 4th innermost contour line of only  $(x, t)$  or  $(y, t)$ ).

**Theorem 3 (Min, sum and preferential attachment kernel contain a renormalised graph sequence).** *Suppose  $\gamma > \frac{\delta}{\delta+1}$  and the kernel is  $g^{min}$ ,  $g^{sum}$  or  $g^{pa}$ . Then if  $\beta > \beta_c$  the infinite connected component contains a graph renormalized for the sequence  $C_n = (n + 1)^{2d}$  almost surely. Consequently, the infinite component is transient.*

### 3.2 Transience in the Non-robust Case

Similar to the robust case, transience in the non-robust case requires a renormalisation argument. However, unlike in the previous section, the paths which carry the random walk out of any finite neighbourhood of  $\mathbf{0}$  are not supported by vertices of extremely large weight which in turn are incident to very long edges, independently of the overall density of edges. Instead, the walk travels along a multitude of moderately long edges; if  $\delta < 2$  and  $\beta > \beta_c$  then there are sufficiently many of these edges and the walk is transient. The reason behind this difference is the same structural feature that distinguishes robustness from non-robustness: In the robust phase, there is a backbone of very few hubs of extremely high weight that guarantees a high connectivity of the network, whereas in the non-robust phase, these hubs are absent and high connectivity can only be obtained by strongly amplifying the edge density.

We have seen in Sect. 3.1 that the precise way of forming the connections between vertices of large weight in the robust case depends on the form of the kernel function  $g$ . In the non-robust case, the form of the kernel  $g$  is much less

important for the proof of transience, and only requires that the profile decays sufficiently slowly. More precisely, we require only the existence of  $s_*$  such that

$$\liminf_{v \rightarrow \infty} \rho(g(s_*, s_*) v^d) v^{\delta d} > 0, \tag{14}$$

for some  $\delta < 2$ . As soon as (14) is satisfied, any supercritical weight-dependent random connection model contains a sub-graph that shows the same qualitative behaviour as a supercritical cluster in long-range percolation with tail exponent  $\delta$ , which is known to be transient [2].

**Theorem 4 (Non-robust supercritical clusters are transient if  $\delta < 2$ ).** *Let  $\mathcal{G} = \mathcal{G}_\varphi(\xi)$  denote the weight-dependent random connection model with  $\rho, g$  satisfying (14) for some  $\delta < 2$ . If  $\mathcal{G}$  is supercritical, then the infinite cluster is transient.*

For a detailed proof of Theorem 4 we refer the reader to the full version of this paper [10]. Here, we provide a condensed version of our argument and briefly discuss its limitations. To relate  $\mathcal{G}$  to long-range percolation, we use a coarse graining technique.  $\mathbb{R}^d$  is partitioned into cubes and these cubes form the sites in a *long-range bond-site percolation model*. Connectivity between sites is established using the edge-probabilities inherited from the underlying weight-dependent random connection model. The crucial ingredient is that, for a site to be *present at all* in the coarse grained model, the corresponding cube needs to contain a cluster of  $\mathcal{G}$  that is sufficiently dense in the cube.

**Proposition 2 (Local density of clusters).** *Let  $\mathcal{G}$  be as in Theorem 4. For any  $\lambda \in (0, 1)$ , and any  $\varepsilon > 0$ , there is a sufficiently large  $M_0 \in \mathbb{N}$ , such that the following is true for all  $M > M_0$ : the probability that the cube  $[0, M]^d$  contains a cluster with at least  $M^{\lambda d}$  vertices exceeds  $1 - \varepsilon$ .*

The proof of Proposition 2 is precisely where the renormalisation scheme mentioned above comes into play. Just as in the robust case, vertices are grouped into boxes on the initial scale. Boxes are called *good* if the vertices inside a box form sufficiently large clusters, and *bad* if this is not the case. On all subsequent scales, boxes themselves are grouped into larger boxes. The larger boxes are in turn good if they have many good sub-boxes and the clusters inside these good sub-boxes are sufficiently well-connected with each other, and thus form a single cluster on a larger scale. When proceeding upwards in this hierarchy, one needs to control the probability that several sub-clusters inside a box do not belong to a single larger cluster. Our estimate for this probability is obtained from an auxiliary construction describing how clusters inside boxes merge. To formulate it, let  $\delta_0 \in (1, 2)$ ,  $\mathbf{m} = (m_1, \dots, m_r)$  with  $r \in \mathbb{N}$  and  $m_j \in \mathbb{N}, j = 1, \dots, r$  and let  $\mathcal{I}_{\mathbf{m}, \delta_0}$  denote the *inhomogeneous random graph* which is constructed on the vertex set  $\{1, \dots, r\}$  by creating edges between  $1 \leq i < j \leq r$  independently with probability

$$1 - e^{-m_i m_j / (\sum_{k=1}^r m_k)^{\delta_0}}.$$

It is instructive to interpret  $\mathbf{m}$  as mass distribution and  $|\mathbf{m}| := \sum_{k=1}^r m_k$  as *total mass* of  $\mathcal{I}_{\mathbf{m}, \delta_0}$ . At any stage of the renormalisation scheme, the clusters inherited

from the previous stage are interpreted as the vertices in an inhomogeneous random graph and the cluster sizes as the corresponding masses. Our probability bounds for the merger of clusters rely on the following result:

**Lemma 2 (Berger, [2, Lemma 2.5]).** *Let  $\delta_0 \in (1, 2)$  and  $\varrho \in (0, 1)$  such that*

$$18\varrho > 16 + \delta_0.$$

*There exist  $\zeta = \zeta(\delta_0, \varrho) > 0$  and  $M_0(\delta_0, \varrho) > 0$  such that for all  $\mathbf{m}$  with  $|\mathbf{m}| \geq M_0$*

$$\mathbb{P}(N_{|\mathbf{m}|e}(\mathcal{I}_{\mathbf{m}, \delta_0}) > 1) < |\mathbf{m}|^{-\zeta},$$

*where  $N_x(\mathcal{I}_{\mathbf{m}, \delta_0})$  denotes the number of clusters  $C \subset \mathcal{I}_{\mathbf{m}, \delta_0}$  with  $\sum_{j \in C} m_j \geq x$ .*

Note that the assumption that  $\delta \leq \delta_0 < 2$  is necessary to apply Lemma 2. This is precisely the reason why neither for the weight-dependent random connection model, nor for any other known long-range percolation model with polynomial tail connection probabilities, the proof of transience in the non-robust case can be extended to the case where  $\delta \geq 2$ .

After invoking Lemma 2, a union bound over all stages of the renormalisation shows that if the scaling parameters are carefully tuned, then the total probability of ever encountering a bad box when zooming outward from  $\mathbf{0}$  can be made arbitrarily small, which implies Proposition 2. In turn, Proposition 2 implies that the site density in the coarse grained model can be brought arbitrarily close to one and thus the transience of the coarse-grained long-range bond-site percolation model follows from the corresponding result for the long-range bond percolation model.

## 4 Recurrence

In order to show recurrence in dimensions  $d \in \{1, 2\}$ , we use electrical network theory.

**Proposition 3 (Nash-Williams, [26]).** *Let  $G$  be a graph with conductance  $C_e$  on every edge  $e$ . Consider a random walk on the graph such that when the particle is at some vertex, it chooses its way with probabilities proportional to the conductances on the edges that it sees. Let  $\{\Pi_n\}_{n=1}^\infty$  be disjoint cut-sets, and denote by  $C_{\Pi_n}$  the sum of the conductances of  $\Pi_n$ . If*

$$\sum_n C_{\Pi_n}^{-1} = \infty,$$

*then the random walk is recurrent.*

The arguments that lead to the result for  $d = 1$  and  $d = 2$  are subtly different from each other, but roughly correspond to showing that the number of edges leaving disjoint cut-sets are sufficiently light tailed. In dimension 1, this can be shown directly by simply treating each edge as having unit conductance

and counting the expected number of edges connecting subsequent cut-sets. In dimension 2, we consider instead a lattice based graph that is constructed so as to have effective conductance that is not smaller than that of our random graph. Then, using a projection argument similar to the one in [2] on this lattice graph, a sufficient condition for recurrence can again be obtained. Putting these arguments together, we obtain the following result.

**Theorem 5 (Recurrence in one and two dimensions, [2]).** *For  $d = 1$  let  $G$  be a random graph on a unit intensity Poisson point process where two vertices  $x$  and  $y$  are connected with probability  $P_{|x-y|}$  such that*

$$\limsup_{v \rightarrow \infty} v^2 P_v < \infty.$$

*For  $d = 2$  let  $G$  be a random graph on a unit intensity Poisson point process where two vertices  $x$  and  $y$  are connected with probability  $P_{|x_1-y_1|, |x_2-y_2|}$  such that*

$$\limsup_{u, v \rightarrow \infty} (u + v)^4 P_{u, v} < \infty.$$

*In both cases, any infinite component of such graph is recurrent.*

Consequently in dimensions 1 and 2, if  $\delta > 2$  and  $\gamma < \delta/(\delta + 1)$  for the preferential attachment, sum and min kernels, any infinite component is recurrent. Similarly, if  $\delta > 2$  and  $\gamma < 1/2$  for the product kernel, any infinite component is recurrent.

**Acknowledgement.** We acknowledge support from DFG through Scientific Network *Stochastic Processes on Evolving Networks*.

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