The Emergence of a Giant Component in One-Dimensional Inhomogeneous Networks with Long-Range Effects

Peter Gracar[1], Lukas Lüchtrath[2](✉), and Christian Mönch[3]

[1] University of Leeds, Leeds, UK
P.Gracar@leeds.ac.uk
[2] Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany
lukas.luechtrath@wias-berlin.de
[3] Johannes Gutenberg Universität Mainz, Mainz, Germany
cmoench@uni-mainz.de

Abstract. We study the weight-dependent random connection model, a class of sparse graphs featuring many real-world properties such as heavy-tailed degree distributions and clustering. We introduce a coefficient, δ_{eff}, measuring the effect of the degree-distribution on the occurrence of long edges. We identify a sharp phase transition in δ_{eff} for the existence of a giant component in dimension $d = 1$.

Keywords: Long-range effects · Percolation · Phase transition · Spatial random graphs · Preferential attachment · Boolean model

1 Introduction and Statement of Result

Complex real-world systems can be seen as a collection of numerous objects interacting with each other in specific ways. This holds in many different contexts and fields such as biology, physics, telecommunications, social sciences, information technology and more. Put differently, many complex systems can be seen as a network where the objects are described by the network's nodes and a link between two nodes in the network indicates the interaction between the corresponding objects. Therefore, over the last 20 years complex networks have become a key tool used to describe real-world systems and related problems. Despite the large amount of uncertainty and complexity arising from their dynamical nature, it is of great importance to understand the structure of the underlying network when analysing a such a system. What kind of phenomena arise in the system and how can they be explained by the way the network is built? These are typical questions in the scientific community but which are also of public interest as their answers may affect decisions made by political or economic leaders.

In recent years, the increase in computing power has made more and more real-world networks amenable to empirical analysis. Most interestingly, despite their different contexts, many such networks have similar structural properties,

M. Dewar et al. (Eds.): WAW 2023, LNCS 13894, pp. 19–35, 2023.
https://doi.org/10.1007/978-3-031-32296-9_2

see e.g. [5,8]. Can these common structural features be explained in a simple way by basic local principles which are shared by the different networks? Such mechanisms are for instance [13]:

- Networks contain nodes that are far more influential than an average node; the so called *hubs* or *stars*.
- Networks show strong *clustering*: Nodes sharing a common neighbour are much more likely to be connected by an edge themselves than nodes that are picked randomly.

Put differently, nodes prefer to connect to similar nodes or to highly influential ones. Despite this, links between nodes that are neither similar nor influential should still occasionally arise. We are interested in models based on these building principles and how classical properties of networks such as the degree distribution, the size of the largest connected component or typical graph distances are affected by them.

In this article, we present a class of models where the vertices are embedded into Euclidean space and each vertex is given an independent weight. Here, the spatial location of a vertex abstractly represents some intrinsic feature and spatially close vertices can be seen as similar and eager to connect to each other. The weight of a vertex represents its influence within the system. The connection mechanism then depends on both weight and spatial distance, hence connections to spatially close vertices or vertices with a large weight are more probable. The introduction of weights guarantees heavy-tailed degree distributions leading to the existence of hubs. The spatial embedding leads to clustering. However, the connection mechanism is set up in a way that still allows far apart vertices having only typical weights to occasionally connect. We call connections of the latter type *long-range* connections. We introduce a coefficient, δ_{eff}, depending only on basic model parameters, which quantifies the overall occurrence of long-range connections in a way that makes it comparable to classical long-range percolation without weights. In this standard model edges are present independently from each other with a probability decaying polynomially in the distance of the endpoints and it corresponds to a homogeneous version of our model. In the present work we focus on the question of existence of a giant component in one dimension, that is, a connected component whose size is of the same order as the entire network. The geometric restrictions of one dimensional space make the existence of a giant rather difficult to achieve. We shall see that the behaviour of long-range percolation is paradigmatic for the models involving weights when expressed in terms of δ_{eff}. Although we focus primarily on one dimension, δ_{eff} also plays a significant role in higher dimensions, for example in providing a sufficient criterion for the existence of subcritical percolation phases [24]. We further believe that it also provides a sufficient criterion for transience of the infinite limit of the giant cluster.

We note that δ_{eff} is a purely theoretical value determined by the network topology that characterises the behaviour of certain useful properties as outlined above. At this stage we are unable to provide any approaches for estimating δ_{eff} using observed data. Any such method would need to take into account that δ_{eff}

inherently involves multiple scales, as it describes the decay of edge probabilities between aggregate vertex sets on increasingly larger scales. This contrasts with the homogeneous case, in which the decay exponent can be estimated directly from the empirical edge length distribution. We believe that estimating δ_{eff} is an interesting statistical problem with potential applications in the analysis of large scale networks.

1.1 The Weight-Dependent Random Connection Model

The *weight-dependent random connection model* is a class of infinite graphs on the points of a Poisson point process on $\mathbb{R}^d \times (0,1)$ that has been intensively studied in recent years [15–17]. In the present article we introduce a finite version of the weight-dependent random connection model constructed on the unit torus $\mathbb{T}_1^d = (-1/2, 1/2)^d$ equipped with torus metric

$$\mathrm{d}_1(x,y) = \min\{|x-y+u| : u \in \{-1,0,1\}^d\}, \text{ for } x,y \in \mathbb{T}_1^d$$

to avoid boundary effects. Here and throughout the paper $|\cdot|$ denotes the Euclidean norm. For $N > 0$ and $\beta > 0$, we construct the graph $\mathscr{G}_N^\beta$ in the following way: The vertex set of $\mathscr{G}_N^\beta$ is a Poisson point process $\mathcal{X}_N$ of intensity N on $\mathbb{T}_1^d \times (0,1)$. We denote a vertex by $\mathbf{x} = (x, t_x) \in \mathcal{X}_N$ and call $x \in \mathbb{T}_1^d$ the *location* and $t_x \in (0,1)$ the *mark* of the vertex. Given $\mathcal{X}_N$, each pair of vertices $\mathbf{x} = (x, t_x)$ and $\mathbf{y} = (y, t_y)$ is connected independently by an edge with probability

$$1 \wedge \left(\tfrac{1}{\beta}(t_x \wedge t_y)^\gamma (t_x \vee t_y)^\alpha N\, \mathrm{d}_1(x,y)^d\right)^{-\delta}, \tag{1}$$

for $\gamma \in [0,1), \alpha \in [0, 2-\gamma)$ and $\delta > 1$ where we denote with $a \wedge b$ the minimum and with $a \vee b$ the maximum of a and b.

Remark 1.

(i) As the typical distance of a point to its nearest neighbour in $\mathcal{X}_N$ is of order $N^{-1/d}$, it is necessary to scale the distance by it to avoid that the graph degenerates. Note that in law it is the same to construct the graph on the points of a unit intensity Poisson process on the volume N torus $\mathbb{T}_N^d = (-N^{1/d}/2, N^{1/d}/2)^d$ and replace $N\,\mathrm{d}_1(x,y)^d$ in (1) by $\mathrm{d}_N(x,y)^d$, the torus metric of $\mathbb{T}_N^d$.

(ii) The parameter $\beta > 0$ controls the edge intensity. Since $x \mapsto 1 \wedge x^{-\delta}$ is a non increasing function, a larger value of β increases the connection probability which then leads to more edges on average.

(iii) The parameters γ and α control the way of influence that vertex marks have on the connection mechanism. By construction, connections to vertices with a small mark are preferred, see also Fig. 2 below. The mark can therefore be seen as the inverse weight of a vertex, giving the model its name. Various choices of γ and α lead to (finite versions of) various models established in the literature, see Table 1.

Table 1. Various choices for γ, α and δ and the models they represent in their infinite version. Here, to shorten notation, $\delta = \infty$ represents models constructed with a function ρ of bounded support, cf. Remark 1(iv).

Parameters	Names and references
$\gamma = 0, \alpha = 0, \delta = \infty$	random geometric graph, Gilbert's disc model [11]
$\gamma = 0, \alpha = 0, \delta < \infty$	random connection model [29,31], long-range percolation [32]
$\gamma > 0, \alpha = 0, \delta = \infty$	Boolean model [12,18], scale-free Gilbert graph [20]
$\gamma > 0, \alpha = 0, \delta < \infty$	soft Boolean model [14]
$\gamma = 0, \alpha > 1, \delta = \infty$	ultra-small scale-free geometric network [33]
$\gamma > 0, \alpha = \gamma, \delta \leq \infty$	scale-free percolation [6,7], geometric inhomogeneous random graphs [4]
$\gamma > 0, \alpha = 1 - \gamma, \delta \leq \infty$	age-dependent random connection model [13]

(iv) The parameter δ controls the occurrence of long edges. The larger the value of δ, the stronger the effect of the geometric embedding is and the less long edges are present. One can replace the function $1 \wedge x^{-\delta}$ in (1) by a non increasing function $\rho : (0, \infty) \to [0, 1]$ and the geometric restrictions become hardest when ρ is of bounded support. Results about such a model can be derived from our model as a limit $\delta \to \infty$.

(v) The restrictions for γ, α and δ guarantee that

$$\int_0^1 \mathrm{d}s \int_t^1 \mathrm{d}t \int_0^\infty \mathrm{d}x \, \left(1 \wedge \left(\beta^{-1} s^\gamma t^\alpha x\right)^{-\delta}\right) < \infty,$$

and therefore all expected degrees remain finite when $N \to \infty$. Consequently, the graph is sparse in the sense that the number of edges is of the same order as the size of the graph.

For all choices of γ, δ and α, the above model converges to a local limit as $N \to \infty$ [13,22], where the limiting graph $\mathscr{G}^\beta = \mathscr{G}^\beta_\infty$ is constructed as follows: The vertex set is given by a unit intensity Poisson process on $\mathbb{R}^d \times (0, 1)$ and two given points $\mathbf{x} = (x, t_x)$ and $\mathbf{y} = (y, t_y)$ are connected independently by an edge with probability

$$1 \wedge \left(\tfrac{1}{\beta}(t_x \wedge t_y)^\gamma (t_x \vee t_y)^\alpha |x - y|^d\right)^{-\delta}. \tag{2}$$

Here, the term local limit is to be understood in the following way: Add a vertex $\mathbf{0} = (0, U)$ at the origin to the graph having a uniform mark U and connect it to all other vertices by rule (2). By Palm theory [27, Chapter 9] this is the same as shifting the graph such that a typical vertex is located at the origin. Then for each event $A(\mathbf{0}, \mathscr{G}^\beta_N)$ depending on the origin and a bounded graph neighbourhood of it in $\mathscr{G}^\beta_N$, we have

$$\lim_{N \to \infty} \mathbb{P}(A(\mathbf{0}, \mathscr{G}^\beta_N)) = \mathbb{P}(A(\mathbf{0}, \mathscr{G}^\beta_\infty)).$$

Put differently, when the number of vertices N tends to infinity, the local neighbourhoods in $\mathscr{G}^\beta_N$ and $\mathscr{G}^\beta_\infty$ look the same.

A similar modelling approach only using a different parametrisation is that of "geometric inhomogeneous random graphs" [4,25] and their infinite volume local limits. All appearing parameters in both approaches can be translated from one model into the other [23].

The limiting graph can then directly be used to derive results for the (asymptotic) degree distribution and clustering since both depend only on graph neighbourhoods of length at most two. The following theorem summarises results from [13,28].

Theorem 1 (Degree distribution and clustering). *Let $\mathscr{G}_N^\beta$ be the weight-dependent random connection model for some choice of $\delta > 1$, $\gamma \in [0,1)$ and $\alpha \in [0, 2-\gamma)$.*

(i) There exists a probability sequence $(\mu_k : k \geq 0)$ such that in probability

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\mathbf{x}\in\mathscr{G}_N^\beta} \mathbb{1}_{\{\mathbf{x} \textit{ has } k \textit{ neighbours in } \mathscr{G}_N^\beta\}} = \mu_k.$$

Moreover, for $\tau := 1 + (1/\gamma \wedge 1/(\gamma+\alpha-1)^+)$, we have

$$\lim_{k\to\infty} k^{\tau+o(1)}\mu_k = 1.$$

(ii) Denote by $V_2(\mathscr{G}_N^\beta)$ the set of vertices with at least two neighbours in $\mathscr{G}_N^\beta$. If $\mathbf{y}$ and $\mathbf{z}$ are neighbours of a vertex $\mathbf{x}$, we call $\{\mathbf{x},\mathbf{y},\mathbf{z}\}$ a triangle when also $\mathbf{y}$ and $\mathbf{z}$ are connected by an edge. Then, there exists a positive constant c depending only on the model parameters such that in probability

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\mathbf{x}\in V_2(\mathscr{G}_N^\beta)} \frac{\sharp\{\textit{triangles containing } \mathbf{x}\}}{\binom{\sharp\{\textit{neighbours of } \mathbf{x}\}}{2}} = c.$$

Theorem 1(i) shows that the degree distribution is heavy-tailed and therefore $\mathscr{G}_N^\beta$ contains the aforementioned hubs. Part (ii) shows that $\mathscr{G}_N^\beta$ indeed exhibits clustering.

From a modelling point of view, the weight-dependent random connection model has the huge advantage that it allows a large flexibility in modelling the way the weight influence the networks geometry. Moreover, sparseness of the graph together with the conditional independence and the ranking of the vertices by their marks can be used to construct the graph in linear time. The following result is an adaption of [3].

Theorem 2. *If $\gamma > 0$, then $\mathscr{G}_N^\beta$ can be sampled in time $O(N)$.*

1.2 Main Result

More difficult than deriving the degree distribution or clustering of a network is the question of the existence of a connected component of linear size. More precisely, we say that $\mathscr{G}_N^\beta$ contains a *giant component* if

$$\lim_{\varepsilon\downarrow 0} \limsup_{N\to\infty} \mathbb{P}(\sharp\mathscr{C}(\mathscr{G}_N^\beta) < \varepsilon N) = 0,$$

where $\mathscr{C}(\mathscr{G}_N^\beta)$ denotes the largest connected component of $\mathscr{G}_N^\beta$ and $\sharp\mathscr{C}(\mathscr{G}_N^\beta)$ the number of vertices within. Note that the existence of a giant component does not only depend on bounded graph neighbourhoods but on the entire graph. Therefore, the local limit structure cannot be used directly. However, it is known that the existence of a giant is highly linked with the existence of an infinite component in the limiting graph. Only when the limiting graph contains an infinite component, a giant component can exist [21]. Our main theorem concerns the existence of a giant component in dimension $d = 1$ where the existence of infinite components in the limit is particularly hard due to the restrictions of the real line $\mathbb{R}$. For the standard long-range percolation model, i.e. the local limit of our model for the choice of $\alpha = \gamma = 0$, it is known that this question depends on the occurrence of long-edges, measured by δ. More precisely, there exists an infinite connected component in the limit for large enough β if $\delta \leq 2$ but there does not exist such a component for all β if $\delta > 2$ [1,30]. We now introduce the *effective decay exponent* δ_{eff}, measuring the influence of the vertex weights on the occurrence of long edges, as

$$\delta_{\text{eff}} := \lim_{n\to\infty} \frac{\log \int_{1/n}^1 \mathrm{d}s \int_s^1 \mathrm{d}t \, \left(1 \wedge (s^\gamma t^\alpha n)^{-\delta}\right)}{\log n}. \tag{3}$$

Theorem 3 (Existence vs. non existence of a giant). *Let $\mathscr{G}_N^\beta$ be the weight-dependent random connection model in dimension $d = 1$.*

(i) If $\delta_{eff} < 2$, then the network $\mathscr{G}_N^\beta$ contains a giant component for large enough values of β.
(ii) If $\delta_{eff} > 2$, then the limiting graph $\mathscr{G}_\infty^\beta$ does not contain an infinite component for any value of β and no giant component can exist in $\mathscr{G}_N^\beta$.

1.3 Examples

In this section we present and further discuss two particularly interesting examples covered by our framework.

Age-Based Spatial Preferential Attachment. This model can be seen as the most natural model in our framework. It corresponds to the choice of $\gamma > 0$ and $\alpha = 1-\gamma$. Its local limit is known under the name *age-dependent random connection model* [13] and it is a type of preferential attachment model. In it, the marks represent the vertices' birth times and early birth times correspond to old hence present for a long time vertices. As $\gamma + \alpha = 1$, one can rewrite the connection probability (1) by

$$1 \wedge \left(\tfrac{1}{\beta}(Nt_x \wedge Nt_y)^\gamma (Nt_x \vee Nt_y)^{1-\gamma} \, \mathrm{d}_1(x,y)^d\right)^{-\delta}$$

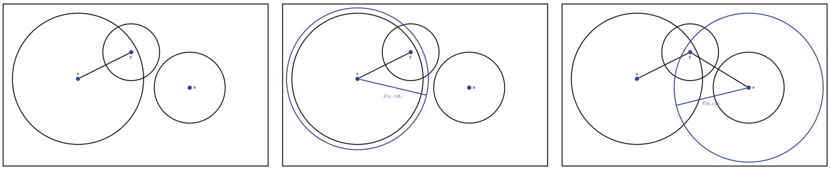

Fig. 1. Example for the connection mechanism of the soft Boolean model in two dimensions. The solid lines represent the graph's edges.

and hence the model can also be constructed on a unit intensity Poisson process on $\mathbb{T}_1^d \times (0, N)$. In this situation, vertices arrive after standard exponential waiting times, justifying the notion of marks being birth times. Since

$$\delta_{\text{eff}} \begin{cases} < 2, & \gamma > 1 - \frac{1}{\delta}, \\ = 2, & \gamma \leq 1 - \frac{1}{\delta} \end{cases},$$

the graph contains a giant component for sufficiently large β when $\gamma > 1 - 1/\delta$. From [16], we can derive that the critical value β_c after which a giant exists is larger than zero for $\gamma < 1 - 1/(\delta+1)$ and zero for $\gamma > 1 - 1/(\delta+1)$. The case $\delta_{\text{eff}} = 2$ is the boundary case which is not covered by our main theorem. However, for $\gamma = 1/2$ the values of γ and α coincide and the minimum and maximum structure in (2) reduces to the product $\sqrt{t_x t_y}$. Therefore, our model coincides with a hyperbolic random graph model [26] for which the existence of a giant is known for large enough β [2]. Hence, by a domination argument, the considered model contains a giant component for all $\gamma \geq 1/2$. It remains an interesting open problem whether this remains true for $\gamma \in (0, 1/2)$.

Soft Boolean Model. This model corresponds to the choice of $\gamma > 0$ and $\alpha = 0$. Following the representation of [14], each vertex x is assigned an independent radius $R_x := t_x^{-\gamma/d}$. Additionally, each potential edge $\{x, y\}$ is assigned an independent random variable $Z(x, y)$ with tail-distribution function $\mathbb{P}(Z(x, y) \geq z) = 1 \wedge z^{-\delta}$. Given the vertices and the collection of $Z(x, y)$, two vertices in $\mathscr{G}_N^\beta$ are connected by an edge, when

$$N^{1/d} \, \mathrm{d}_1(x, y) \leq \beta^{1/d} Z(x, y) \big(R_x \vee R_y\big).$$

That is, the vertices share an edge when in the rescaled picture (cf. Remark 1(i)), the vertex with smaller assigned radius is contained in the ball centered at the stronger vertex with the assigned radius stretched by a heavy-tailed random variable, cf. Fig. 1. For $\delta \to \infty$ one derives a version of the classical Boolean model [20]. We calculate

$$\delta_{\text{eff}} = 1 + \delta(1 - \gamma) \begin{cases} < 2, & \gamma > 1 - \frac{1}{\delta}, \\ = 2, & \gamma = 1 - \frac{1}{\delta}, \\ > 2, & \gamma < 1 - \frac{1}{\delta}. \end{cases}$$

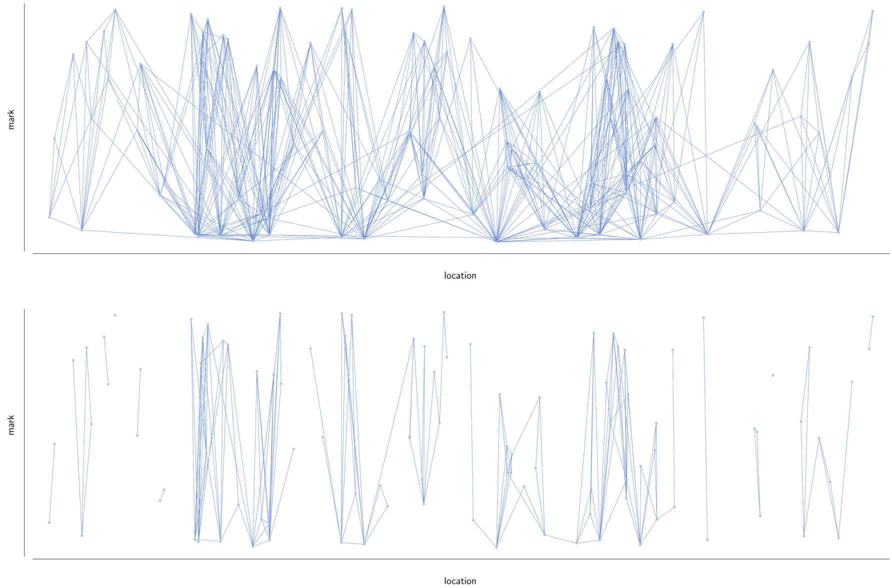

Fig. 2. Two simulations for the (rescaled) soft Boolean model on the torus $\mathbb{T}^1_{100}$ with $\delta = 3.5$ and $\beta = 1$. The first picture was simulated with $\gamma = 0.75$, the second one with $\gamma = 0.51$.

Therefore, the graph $\mathscr{G}_N^\beta$ contains a giant component for large enough values of β for $\gamma > 1 - 1/\delta$ but does not for $\gamma < 1 - 1/\delta$, see also Fig. 2 for some simulations.

2 Proof of the Main Theorem

2.1 Some Construction and Notation

From now on, we work exclusively in dimension $d = 1$. We also work on the rescaled picture based on Remark 1(i). Then the underlying vertex set of our graphs can be constructed in the following way. We start with a vertex $X_0 = 0$ placed at the origin. Let $(Z_i : i \in \mathbb{N})$ and $(\tilde{Z}_i : i \in \mathbb{N})$ be two independent sequences of independent standard exponential random variables. For $i \in \mathbb{N}$ set $X_i = \sum_{j=1}^{i} Z_j$ and for $i \in \mathbb{Z}\backslash\mathbb{N}_0$ set $X_i = -\sum_{j=1}^{|i|} \tilde{Z}_j$. Then $\eta_0 := \{X_i : i \in \mathbb{Z}\}$ is the Palm version [27] of a unit intensity Poisson process on the real line where a distinguished vertex is placed at the origin. We call η_0 the *vertex locations*. Note that we have $X_i < X_j$ whenever $i < j$. Further, let $\mathcal{T}_0 = (T_i : i \in \mathbb{Z})$ be a sequence of independent Uniform$(0,1)$ random variables, independent of η_0. The elements of $\mathcal{T}_0$ are the *vertex marks* and we define

$$\mathcal{X}_0 = \big(\mathbf{X}_i = (X_i, T_i) \in \eta_0 \times \mathcal{T}_0 : i \in \mathbb{Z}\big)$$

the Palm version of the marked Poisson process which is our vertex set. Moreover, define $\mathcal{U}_0 = (U_{i,j} : i < j \in \mathbb{Z})$ another sequence of independent Uniform$(0,1)$ random variables, which we call *edge marks*. Finally define for all $i < j \in \mathbb{Z}$ the random variables

$$E_{i,j} := E_{i,j}(\mathcal{X}_0, \mathcal{U}_0) := \mathbb{1}_{\{U_{i,j} < (\beta^{-1}(T_i \wedge T_j)^{\gamma}(T_i \vee T_j)^{\alpha})|X_i - X_j|)^{-\delta}\}}.$$

Note that the random variables $(E_{i,j} : i < j \in \mathbb{Z})$ are only conditionally independent given $\mathcal{X}_0$ but not in general. The graph $\mathscr{G}^\beta := \mathscr{G}^\beta_\infty(\mathcal{X}_0, \mathcal{U}_0)$ is then defined through its vertex set $\mathcal{X}_0$ and (random) adjacency matrix $(E_{i,j} : i < j \in \mathbb{Z})$.

As a Poisson process remains a Poisson process when restricted to an area, the finite graphs $\mathscr{G}^\beta_N$ can be constructed by the reduction of $\mathcal{X}_0$ and $\mathcal{U}_0$ to the elements in $(-N/2, N/2)$. Note that we may have changed the distance in the connection probability from the torus metric to the Euclidean one. However, Theorem 3(ii) only concerns the infinite model where no change is made and in Theorem 3(i) edges may be removed from the graph due to the larger distances at the boundary which makes the existence of a giant less likely.

Throughout the remaining text, we use the notation $\mathbf{X}_i \sim \mathbf{X}_j$ to denote the event that $\mathbf{X}_i$ and $\mathbf{X}_j$ are connected by an edge. For two sets of vertices V_1, V_2, we write $V_1 \sim V_2$, if they are connected by a direct edge, i.e. there exist $\mathbf{X} \in V_1$ and $\mathbf{Y} \in V_2$ such that $\mathbf{X} \sim \mathbf{Y}$. For two positive functions f and g, we write $f \asymp g$ when f/g is bounded away from zero and infinity. We write $f = o(g)$ if $f(x)/g(x) \to 0$ and $f = O(g)$ if $f(x)/g(x) \to C < \infty$ when $x \to \infty$.

Finally, we say that an event A is *increasing*, if the function $\mathbb{1}_A$ increases whenever additional vertices are added to the graph, vertex marks are decreased (and hence the weights are increased) or edge marks are decreased (and hence potentially more edges are added). For two such events A and B, the FKG-inequality [10] in a version of [19] yields

$$\mathbb{P}(A \cap B) = \mathbb{E}[\mathbb{P}(A \cap B \mid \eta)] \geq \mathbb{E}[\mathbb{P}(A \mid \eta)\mathbb{P}(B \mid \eta)] \geq \mathbb{P}(A)\mathbb{P}(B). \quad (4)$$

2.2 Connecting Far Apart Vertex Sets

To understand the role of δ_{eff}, one has to understand the influence of the vertex marks to the occurrence of long edges. These are essential for large components in dimension one as they are needed to overcome 'bad regions' where far less than average vertices and edges are placed. For large n, the minimum of n independent uniform random variables is roughly $1/n$ and consequently the double integral appearing in (3) is essentially the probability of two randomly picked vertices from vertex sets of size n at distance roughly n being connected by an edge. Ignoring additional correlations between edges and treating the random variables $E_{i,j}$ as independent, the expected number of edges is then roughly given by $n^{2-\delta_{\text{eff}}}$ which grows large for $\delta_{\text{eff}} < 2$ but vanishes for $\delta_{\text{eff}} > 2$. This heuristic is justified in the following lemma which is formulated for one dimension but can be generalised to higher dimensions. For its formulation we define the two sets of vertices

$$V_\ell^n := \{\mathbf{X}_{-2n}, \dots, \mathbf{X}_{-n-1}\} \quad \text{and} \quad V_r^n := \{\mathbf{X}_n, \dots, \mathbf{X}_{2n-1}\}$$

which play the role of the two sets of size n at distance roughly n.

Lemma 1.

(i) For each $\varepsilon > 0$*, there exists* $\mu \in (0, 1/2)$ *and a constant* $C > 0$ *such that*

$$\mathbb{P}(V_\ell^n \not\sim V_r^n) \le \exp\big(-Cn^{2-\delta_{\mathrm{eff}}-\varepsilon}\big) + O\big(n^{1-\mu}e^{-n^\mu}\big).$$

(ii) For each $\varepsilon > 0$*, there exists* $\mu \in (0, 1/2)$ *and constants* $C_1, C_2 > 0$ *such that*

$$\mathbb{P}(V_\ell^n \not\sim V_r^n) \ge C_1(1 - n^{-\mu}) \exp\big(-C_2 n^{2-\delta_{\mathrm{eff}}+\varepsilon}\big)$$

Proof. We start with the proof of (i). To control the influence of the vertex marks and the random distances, we rely on some 'regular' behaviour of the underlying point process. Let $\mu \in (0, 1/2)$ and define $N_\ell^n(i) := \sum_{j=-n-1}^{-2n} \mathbb{1}_{\{T_j \le i/\lfloor n^{1-\mu} \rfloor\}}$. We say that V_ℓ^n has *μ-regular marks* if

$$N_\ell^n(i) \ge \frac{in}{2\lfloor n^{1-\mu} \rfloor}, \quad \text{for all } i = 1, \dots, \lfloor n^{1-\mu} \rfloor.$$

Using a standard Chernoff bound for independent uniforms together with the union bound, we infer

$$\mathbb{P}(V_\ell^n \text{ is not } \mu\text{-regular}) = O(n^{1-\mu}e^{-n^\mu}).$$

The same holds verbatim for V_r^n. Moreover, standard large deviation results for the sum of independent exponential random variables yields

$$\mathbb{P}(|X_{2n-1} - X_{-2n}| > 5n) \le e^{-\text{const } n}.$$

Therefore, writing $\tilde{\mathbb{P}}$ for $\mathbb{P}$ conditioned of μ-regular marks in V_ℓ^n and V_r^n as well as $|X_{2n-1} - X_{-2n}| \le 5n$, we have

$$\mathbb{P}(V_\ell^n \not\sim V_r^n) \le \tilde{\mathbb{P}}(V_\ell^n \not\sim V_r^n) + O(n^{1-\mu}e^{-n^\mu}). \tag{5}$$

To calculate the conditional probability in (5), observe that for all given $\mathbf{X}_i \in V_\ell^n$ and $\mathbf{X}_j \in V_r^n$, we have, writing $\rho(x) = 1 \wedge x^{-\delta}$,

$$\tilde{\mathbb{P}}(E_{i,j} = 0 \mid \mathbf{X}_i, \mathbf{X}_j) \le \exp\big(-\rho\big(\beta^{-1}(T_i \wedge T_j)^\gamma (T_i \vee T_j)^\alpha 5n\big)\big).$$

Therefore, by writing F_i^n for the empirical distributions of the vertex marks in V_i^n, for $i = \ell, r$, we infer

$$\begin{aligned}
\tilde{\mathbb{P}}(V_\ell^n \not\sim V_r^n) &= \tilde{\mathbb{E}}\Big[\prod_{i=-n-1}^{-2n} \prod_{j=n}^{2n-1} \tilde{\mathbb{P}}(E_{i,j} = 0 \mid \mathcal{X}_0)\Big] \\
&\le \tilde{\mathbb{E}}\Big[\exp\Big(-\sum_{i=-n-1}^{-2n} \sum_{j=n}^{2n-1} \rho\big(\beta^{-1}(T_i \wedge T_j)^\gamma (T_i \vee T_j)^\alpha 5n\big)\Big)\Big] \\
&= \tilde{\mathbb{E}}\Big[\exp\Big(\int_0^1 F_\ell^n(\mathrm{d}t) \int_0^1 F_r^n(\mathrm{d}s)\ \rho\big(\beta^{-1}(t \wedge s)^\gamma (t \vee s)^\alpha 5n\big)\Big)\Big].
\end{aligned}$$

By μ-regularity, we get by [17, Eq. (8)] that $nF_i^n(t) \geq \frac{n}{3}(t - n^{\mu-1})$ and therefore by a change of variables in the last integral we derive for some constant $C > 0$

$$\begin{aligned}\tilde{\mathbb{P}}(V_\ell^n \not\sim V_r^n) &\leq \exp\Big(-Cn^2 \int_{n^{\mu-1}}^{1-n^{\mu-1}} \mathrm{d}t \int_t^{1-n^{\mu-1}} \mathrm{d}s \, \big(1 \wedge (\tfrac{5}{\beta} t^\gamma s^\alpha n)^{-\delta}\big)\Big) \\ &\leq \exp\big(-Cn^{2-\delta_{\mathrm{eff}}-\varepsilon}\big),\end{aligned}$$

where the last step follows by the fact that the order of the integral is driven by the lower integration bound together with the continuity of the integral in μ. This concludes the proof of (i).

The proof of (ii) works similarly. However, the definition of μ-regularity has to be slightly changed. We now say the marks of V_ℓ^n are μ-regular if

(a) $\min_{-2n \leq j \leq -n-1} T_j \geq \lceil n^{-1-\mu} \rceil$ and

(b) $\sum_{j=-n-1}^{-2n} \mathbb{1}_{\{T_j \leq i/\lceil n^{1-\mu} \rceil\}} \leq \frac{2in}{\lceil n^{1-\mu} \rceil}$

which holds with a probability of order $1 - n^{-\mu}$. Using now a lower bound on the distances and performing similar calculations as above yields (ii), cf. [17, Lemma 4.1].

2.3 Existence of a Giant Component

In this section, we use a renormalisation scheme introduced by Duminil-Copin et al. [9] for the existence of an infinite component in one-dimensional long-range percolation on the lattice to construct a component growing linear with a subsequence of $(\mathscr{G}_N^\beta : N > 0)$ from which we derive the existence of a giant component for large enough values of β whenever $\delta_{\mathrm{eff}} < 2$.

We start by defining the scales on which the renormalisation works. For some $K \in \mathbb{N}$, define $(K_n : n \in \mathbb{N})$ by $K_n := (n!)^3\, K^n$. Define on each scale the blocks of vertices

$$B_{K_n}^i := \{\mathbf{X}_{K_n(i-1)}, \dots, \mathbf{X}_{K_n i}, \dots, \mathbf{X}_{K_n(i+1)-1}\}, \quad i \in \mathbb{Z},$$

and we abbreviate $B_K = B_K^0$. Each of the scale n blocks consists of $2K_n$ vertices and two consecutive blocks intersect on half of their points. We fix a $\vartheta > 3/4$ and say that the block $B_{K_n}^i$ is ϑ-*good* if it contains a connected component of density at least ϑ. Otherwise, we say the block is ϑ-*bad*. Note that due to the overlapping property, the largest components of two consecutive ϑ-good blocks intersect in at least one vertex.

Consider for some $\vartheta > 3/4$ the sequence $\vartheta_n := \vartheta - 2/(n^3\, K)$, where K is chosen large enough to guarantee $\inf_n \vartheta_n > 3/4$. We want to bound the probability of the scale n block B_{K_n} being ϑ_n-bad and we consider the scale $n-1$ blocks $B_{K_{n-1}}^{-n^3 K-1}, \dots, B_{K_{n-1}}^{n^3 K-1}$ contained in it. If all these blocks are ϑ_{n-1}-good so is B_{K_n} by our choice of $\inf_m \vartheta_m > 3/4$. Therefore, either there must exist at least two disjoint ϑ_{n-1}-bad scale $n-1$ blocks or there is one ϑ_{n-1}-bad block $B^i_{K_{n-1}}$ and all blocks disjoint from it are good. The first event is bounded by

$$\mathbb{P}(\exists \text{ two disjoint } \vartheta_{n-1}\text{-bad blocks}) \leq 2(n^3\,K)^2\mathbb{P}(B_{K_{n-1}} \text{ is } \vartheta_{n-1}\text{-bad})^2. \quad (6)$$

We denote the latter event by A_i and have

$$\begin{aligned}\mathbb{P}(\{B_{K_n} \text{ is } \vartheta_n\text{-bad}\} \cap A_i)\\ \leq \mathbb{P}(B_{K_{n-1}} \text{ is } \vartheta_{n-1}\text{-bad}) \sum_{|i|=0}^{n^3\,K-1} \mathbb{P}(B_{K_n} \text{ is } \vartheta_n\text{-bad} \mid A_i).\end{aligned} \quad (7)$$

To calculate the conditional probability, observe that $|i| \notin \{n^3\,K-2, n^3\,K-1\}$ since otherwise B_{K_n} would be ϑ_n-good by our choice of $\vartheta_n = \vartheta_{n-1} - {}^2\!/\!_{(n^3\,K)}$. Hence, by the overlapping property, there exists a connected component $\mathscr{C}^i_\ell$ left of the bad block and a connected component $\mathscr{C}^i_r$ on the right, both of density at least ϑ_{n-1}. Further, if both these clusters are connected by an edge, the whole block B_{K_n} again is ϑ_n-good. Hence,

$$\sum_{|i|=0}^{n^3\,K-1} \mathbb{P}(B_{K_n} \text{ is } \vartheta_n\text{-bad} \mid A_i) \leq \sum_{|i|=0}^{n^3\,K-3} \mathbb{P}(\mathscr{C}^i_\ell \not\sim \mathscr{C}^i_r \mid A_i).$$

Let $\vartheta^* := \inf_m \vartheta_m (> {}^3\!/\!_4)$ and define the 'leftmost'and'rightmost' vertices of B_{K_n} by

$$V^n_\ell(\vartheta^*) := \{\mathbf{X}_{-K_n}, \dots, \mathbf{X}_{-K_n + \lfloor \vartheta^* K_{n-1} \rfloor - 1}\} \quad \text{and}$$
$$V^n_r(\vartheta^*) := \{\mathbf{X}_{K_n - \lfloor \vartheta^* K_{n-1} \rfloor}, \dots, \mathbf{X}_{K_n - 1}\}.$$

We claim $\mathbb{P}(\mathscr{C}^i_\ell \not\sim \mathscr{C}^i_r \mid A_i) \leq \mathbb{P}(V^n_\ell(\vartheta^*) \not\sim V^n_r(\vartheta^*))$, see Lemma 2 below. Combining this with (6) and (7), we infer for all $n \geq 2$

$$\begin{aligned}&\mathbb{P}(B_{K_n} \text{ is } \vartheta_n\text{-bad})\\ &\quad\leq \mathbb{P}(B_{K_{n-1}} \text{ is } \vartheta_{n-1}\text{-bad})\big(2n^3\,K\,\mathbb{P}(V^n_\ell(\vartheta^*) \not\sim V^n_r(\vartheta^*))\big)\\ &\qquad + 2(n^3\,K)^2\,\mathbb{P}(B_{K_{n-1}} \text{ is } \vartheta_{n-1}\text{-bad})^2\\ &\quad\leq \mathbb{P}(B_{K_{n-1}} \text{ is } \vartheta_{n-1}\text{-bad})\big(2n^3\,K\big(\exp(-CK_{n-1}^{2-\delta_{\text{eff}}-\varepsilon}) + O(K_{n-1}^{1-\mu} e^{-K^\mu_{n-1}})\big)\big)\\ &\qquad + 2(n^3\,K)^2\mathbb{P}(B_{K_{n-1}} \text{ is } \vartheta_{n-1}\text{-bad})^2\\ &\quad\leq \frac{1}{100}\mathbb{P}(B_{K_{n-1}} \text{ is } \vartheta_{n-1}\text{-bad}) + 2(n^3\,K)^2\mathbb{P}(B_{K_{n-1}} \text{ is } \vartheta_{n-1}\text{-bad})^2\end{aligned} \quad (8)$$

for sufficiently large K and the right choice of ε and μ using $\delta_{\text{eff}} < 2$. The second inequality follows from Lemma 1(i) and the fact that $K^{\mu-1}_{n-1} \leq K^{\mu-1+\varepsilon'}_n$ for large enough K.

To deal with the first scale, we condition on $|X_{-K} - X_{K-1}| < 5K$ which holds with an error term exponentially small in K. We choose $\beta > 5K$ so that for all pairs of vertices $\mathbf{X}, \mathbf{Y}$ of B_K, on this event, we have $\beta^{-1}(T_x \wedge T_y)^\gamma (T_x \vee T_y)^\alpha |X - Y| < 1$ and therefore both vertices are connected by an edge by connection rule (2). Hence, on this event, the subgraph B_K is complete. Combining this with (8), we infer inductively for all n and large enough K

$$\mathbb{P}(B_{K_n} \text{ is } \vartheta_n\text{-bad}) \le \frac{1}{400(n^3\,K)^2} \le \frac{1}{2}.$$

Now observe, that B_{K_n} is contained in the interval $(-2K_n, 2K_n)$ with an error term exponentially small in K_n and therefore uniformly

$$\mathbb{P}(\sharp\mathscr{C}(\mathscr{G}^{\beta}_{2K_n}) \ge \tfrac{3}{8}K_n) \ge \frac{3}{8}$$

since $\vartheta_n \ge \vartheta^* > 3/4$. We have hence shown that for $\delta_{\mathrm{eff}} < 2$ and large enough K and β, the largest connected component of the subsequence $(\mathscr{G}^{\beta}_{2K_n} : n \in \mathbb{N})$ grows linearly in time. The existence of a giant component for the whole sequence $(\mathscr{G}^{\beta}_{N} : N \in \mathbb{N})$ is then simply a consequence of the ergodicity in our model, cf. [17, Corollary 2.6].

It remains to prove the lemma used in the bound of (8).

Lemma 2. *For all* $|i| \in \{0, \dots, n^3\,K - 3\}$, *we have*

$$\mathbb{P}(\mathscr{C}^i_\ell \not\sim \mathscr{C}^i_r \mid A_i) \le \mathbb{P}(V^n_\ell(\vartheta^*) \not\sim V^n_r(\vartheta^*)).$$

Proof. To shorten notation, we abbreviate $V_\ell = V^n_\ell(\vartheta^*)$ and $V_r = V^n_r(\vartheta^*)$. The proof is based on the idea that belonging to the largest clusters in good boxes gives negative information for not being connected compared to the uniform (i.e. independently sampled) case. Observe first that on the event A_i, we have $\sharp\mathscr{C}^i_\ell \ge \sharp V_\ell$ and $\sharp\mathscr{C}^i_r \ge \sharp V_r$. Let I_ℓ be the (random) set of all indices belonging to the vertices of $\mathscr{C}^i_\ell$ ordered from smallest absolute value to largest and let I_r be the same set for the indices of $\mathscr{C}^i_r$. Let further be $\mathcal{J}_\ell$ a set of $\sharp V_r$-many indices chosen independently from everything else and uniformly among all indices of vertices in B_{K_n} left of the block $B^i_{K_{n-1}}$ and $\mathcal{J}_r$ be the same but for the indices on the right side. Note that due to our construction, the indices are deterministically given. To bound the probability of $\mathscr{C}^i_\ell$ and $\mathscr{C}^i_r$ not being connected by an edge, we first choose a subset of smaller size $\sharp V_\ell = \sharp V_r$ uniformly from both clusters and only ask that there is no edge connecting these. Note that choosing uniform a subset of size $\sharp V_\ell$ from I_ℓ is the same as using the indices in $\mathcal{J}_\ell$, conditioned on $\mathcal{J}_\ell \subset I_\ell$. We infer

$$\begin{aligned}
&\mathbb{P}(\mathscr{C}^i_\ell \not\sim \mathscr{C}^i_r \mid A_i)\\
&\quad= \mathbb{P}\Big(\bigcap_{i_\ell \in I_\ell}\bigcap_{i_r \in I_r}\{E_{i_\ell,i_r} = 0\} \,\Big|\, A_i\Big)\\
&\quad\le \frac{\mathbb{P}\Big(\{\mathcal{J}_\ell \subset I_\ell\} \cap \{\mathcal{J}_r \subset I_r\} \cap \big(\bigcap_{i_\ell \in \mathcal{J}_\ell}\bigcap_{i_r \in \mathcal{J}_r}\{E_{i_\ell,i_r} = 0\}\big) \,\Big|\, A_i\Big)}{\mathbb{P}(\{\mathcal{J}_\ell \subset I_\ell\} \cap \{\mathcal{J}_r \subset I_r\} \mid A_i)}.
\end{aligned}$$

Since $\vartheta^* > 3/4$, on the event A_i, the clusters $\mathscr{C}^i_\ell$ and $\mathscr{C}^i_r$ are the unique largest clusters on the left and right of the block $B^i_{K_{n-1}}$. Therefore, $\{\mathcal{J}_\ell \subset I_\ell\} \cap \{\mathcal{J}_r \subset I_r\}$

is an increasing event, since strengthening the vertices or adding more edges increases the clusters. Note that adding additional vertices to $\mathcal{X}_0$ is equivalent to bringing the vertices closer together which may then also lead to additional edges. Conversely, the event $\{E_{i,j} = 0\}$ is a decreasing event in the sense that $-\mathbb{1}_{\{E_{i,j}=0\}}$ is increasing. Therefore, the FKG-inequality (4) yields

$$\mathbb{P}(\mathscr{C}_\ell^i \not\sim \mathscr{C}_r^i \mid A_i) \le \mathbb{P}\Big(\bigcap_{i_\ell \in \mathcal{J}_\ell} \bigcap_{i_r \in \mathcal{J}_r} \{E_{i_\ell, i_r} = 0\} \,\Big|\, A_i \Big).$$

Moreover, the existence of edges between vertices outside $B^i_{K_n}$ does not depend on the vertices and edges within this block. Denoting by $\tilde{A}_i$ the *increasing* event that all blocks disjoint from $B^i_{K_n}$ are good, another application of the FKG-inequality yields

$$\mathbb{P}(\mathscr{C}_\ell^i \not\sim \mathscr{C}_r^i \mid A_i) \le \mathbb{P}\Big(\bigcap_{i_\ell \in \mathcal{J}_\ell} \bigcap_{i_r \in \mathcal{J}_r} \{E_{i_\ell, i_r} = 0\} \,\Big|\, \tilde{A}_i \Big) \le \mathbb{P}\Big(\bigcap_{i_\ell \in \mathcal{J}_\ell} \bigcap_{i_r \in \mathcal{J}_r} \{E_{i_\ell, i_r} = 0\} \Big).$$

The vertices on the right-hand side are now chosen uniformly at random from all vertices on the left resp. on the right of the bad block. The proof finishes with the observation that in each such sample all vertices and edges have independent and identically distributed marks so that the probability is increased when choosing the left-most resp. right-most vertices maximising the distances between the involved vertices.

2.4 Absence of an Infinite Component

The proof of non-existence of an infinite component for all β when $\delta_{\text{eff}} > 2$ relies on an edge counting argument. We say an edge *crosses the origin* if it connects a vertex left of the origin with one of the right. Here, without loss of generality, we consider $\mathbf{X}_0$ being right of the origin. We show that with a positive probability no such crossing exists. By ergodicity this then holds true for edges crossing any natural number and each component must be finite.

Define for each $n \in \mathbb{N}$ the disjoint sets

$$\begin{aligned} \Gamma_n^\ell &:= \{\mathbf{X}_{-2^n, \dots, \mathbf{x}_{-1}}\}, \quad \Gamma_n^{\ell\ell} := \{\mathbf{X}_{-2^{n+1}}, \dots, \mathbf{X}_{-2^n-1}\}, \\ \Gamma_n^r &:= \{\mathbf{X}_0, \dots, \mathbf{X}_{2^n-1}\}, \Gamma_n^{rr} := \{\mathbf{X}_{2^n}, \dots, \mathbf{X}_{2^{n+1}-1}\}. \end{aligned}$$

We say a crossing of the origin occurs at stage

$n = 1$, if any edge connects the set $\Gamma_1^\ell \cup \Gamma_1^{\ell\ell}$ and $\Gamma_1^r \cup \Gamma_1^{rr}$ or at stage
$n \ge 2$, if any edge connects either $\Gamma_n^{\ell\ell}$ and Γ_n^{rr} or $\Gamma_n^{\ell\ell}$ and Γ_n^{rr} or Γ_n^ℓ and Γ_n^{rr}. Note that any edges connecting Γ_n^ℓ and Γ_n^r have already been considered at an earlier stage.

Let $\chi(n) \in \{0, 1\}$ denote the indicator of the event that there is a crossing occurring at stage n. Since the events $\{\chi(n) = 0\}$ are all decreasing and thus positively correlated, we have

$$\mathbb{P}\Big(\bigcap_{n \in \mathbb{N}} \{\chi(n) = 0\} \Big) \ge \prod_{n \in \mathbb{N}} \mathbb{P}(\chi(n) = 0).$$

Since the product of the right-hand-side is strictly larger than zero if and only if the sum of the probabilities of the complementary events $\{\chi(n) = 1\}$ converges, the proof finishes by applying Lemma 1(ii) and $\delta_{\text{eff}} > 2$ to

$$\sum_{n\in\mathbb{N}} \mathbb{P}(\chi(n) = 1) \le \mathbb{P}(\chi(1) = 1) + 3 \sum_{n\ge 2} \mathbb{P}(\Gamma_n^{\ell\ell} \sim \Gamma_n^{r}).$$

Acknowledgement. We gratefully received support by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - grant no. 443916008 (SPP 2265) and by the Leibniz Association within the Leibniz Junior Research Group on *Probabilistic Methods for Dynamic Communication Networks* as part of the Leibniz Competition. We would also like to thank Arne Grauer who provided us the R-code we used for Fig. 2.

References

1. Aizenman, M., Newman, C.M.: Discontinuity of the percolation density in one-dimensional $1/|x-y|^2$ percolation models. Comm. Math. Phys. **107**(4), 611–647 (1986). http://projecteuclid.org/euclid.cmp/1104116233
2. Bode, M., Fountoulakis, N., Müller, T.: On the largest component of a hyperbolic model of complex networks. Electron. J. Combin. **22**(3), Paper 3.24, 46 (2015). https://doi.org/10.37236/4958
3. Bringmann, K., Keusch, R., Lengler, J.: Sampling geometric inhomogeneous random graphs in linear time. In: 25th European Symposium on Algorithms, LIPIcs. Leibniz International Proceedings in Informatics, vol. 87, pp. Art. No. 20, 15. Schloss Dagstuhl. Leibniz-Zent. Inform. Wadern (2017)
4. Bringmann, K., Keusch, R., Lengler, J.: Geometric inhomogeneous random graphs. Theoret. Comput. Sci. **760**, 35–54 (2019). https://doi.org/10.1016/j.tcs.2018.08.014
5. Chung, F., Lu, L.: Complex graphs and networks, CBMS Regional Conference Series in Mathematics, vol. 107. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (2006). https://doi.org/10.1090/cbms/107
6. Deijfen, M., van der Hofstad, R., Hooghiemstra, G.: Scale-free percolation. Ann. Inst. Henri Poincaré Probab. Stat. **49**(3), 817–838 (2013). https://doi.org/10.1214/12-AIHP480
7. Deprez, P., Wüthrich, M.V.: Scale-free percolation in continuum space. Commun. Math. Stat. **7**(3), 269–308 (2018). https://doi.org/10.1007/s40304-018-0142-0
8. Dorogovtsev, S.N.: Lectures on Complex Networks. Oxford Master Series in Physics, vol. 20. Oxford University Press, Oxford (2010). https://doi.org/10.1093/acprof:oso/9780199548927.001.0001. Oxford Master Series in Statistical Computational, and Theoretical Physics
9. Duminil-Copin, H., Garban, C., Tassion, V.: Long-range models in 1D revisited (2020)
10. Fortuin, C.M., Kasteleyn, P.W., Ginibre, J.: Correlation inequalities on some partially ordered sets. Comm. Math. Phys. **22**, 89–103 (1971). http://projecteuclid.org/euclid.cmp/1103857443
11. Gilbert, E.N.: Random plane networks. J. Soc. Indust. Appl. Math. **9**, 533–543 (1961)

12. Gouéré, J.B.: Subcritical regimes in the Poisson Boolean model of continuum percolation. Ann. Probab. **36**(4), 1209–1220 (2008). https://doi.org/10.1214/07-AOP352
13. Gracar, P., Grauer, A., Lüchtrath, L., Mörters, P.: The age-dependent random connection model. Queueing Syst. **93**(3-4), 309–331 (2019). https://doi.org/10.1007/s11134-019-09625-y
14. Gracar, P., Grauer, A., Mörters, P.: Chemical distance in geometric random graphs with long edges and scale-free degree distribution. Comm. Math. Phys. **395**(2), 859–906 (2022). https://doi.org/10.1007/s00220-022-04445-3
15. Gracar, P., Heydenreich, M., Mönch, C., Mörters, P.: Recurrence versus transience for weight-dependent random connection models. Electron. J. Probab. **27**, 1–31 (2022). https://doi.org/10.1214/22-EJP748
16. Gracar, P., Lüchtrath, L., Mörters, P.: Percolation phase transition in weight-dependent random connection models. Adv. Appl. Probab. **53**(4), 1090–1114 (2021). https://doi.org/10.1017/apr.2021.13
17. Gracar, P., Lüchtrath, L., Mönch, C.: Finiteness of the percolation threshold for inhomogeneous long-range models in one dimension (2022). https://doi.org/10.48550/ARXIV.2203.11966. https://arxiv.org/abs/2203.11966
18. Hall, P.: On continuum percolation. Ann. Probab. **13**(4), 1250–1266 (1985). http://links.jstor.org/sici?sici=0091-1798(198511)13:4⟨1250:OCP⟩2.0.CO;2-U&origin=MSN
19. Heydenreich, M., van der Hofstad, R., Last, G., Matzke, K.: Lace expansion and mean-field behavior for the random connection model (2020)
20. Hirsch, C.: From heavy-tailed Boolean models to scale-free Gilbert graphs. Braz. J. Probab. Stat. **31**(1), 111–143 (2017). https://doi.org/10.1214/15-BJPS305
21. van der Hofstad, R.: The giant in random graphs is almost local (2021). https://doi.org/10.48550/ARXIV.2103.11733. https://arxiv.org/abs/2103.11733
22. van der Hofstad, R., van der Hoorn, P., Maitra, N.: Local limits of spatial inhomogeneous random graphs (2021). https://doi.org/10.48550/ARXIV.2107.08733. https://arxiv.org/abs/2107.08733
23. van der Hofstad, R., van der Hoorn, P., Maitra, N.: Scaling of the clustering function in spatial inhomogeneous random graphs (2022). https://doi.org/10.48550/ARXIV.2212.12885. https://arxiv.org/abs/2212.12885
24. Jahnel, B., Lüchtrath, L.: Existence of subcritical percolation phases for generalised weight-dependent random connection models (2023). https://doi.org/10.48550/ARXIV.2302.05396. https://arxiv.org/abs/2302.05396
25. Komjáthy, J., Lapinskas, J., Lengler, J.: Penalising transmission to hubs in scale-free spatial random graphs. Ann. Inst. Henri Poincaré Probab. Stat. **57**(4), 1968–2016 (2021). https://doi.org/10.1214/21-AIHP1149
26. Komjáthy, J., Lodewijks, B.: Explosion in weighted hyperbolic random graphs and geometric inhomogeneous random graphs. Stoch. Process. Appl. **130**(3), 1309–1367 (2020). https://doi.org/10.1016/j.spa.2019.04.014. https://www.sciencedirect.com/science/article/pii/S0304414918301601
27. Last, G., Penrose, M.: Lectures on the Poisson Process. Cambridge University Press, Cambridge (2017). https://doi.org/10.1017/9781316104477
28. Lüchtrath, L.: Percolation in weight-dependent random connection models. Ph.D. thesis, Universität zu Köln (2022). https://kups.ub.uni-koeln.de/64064/
29. Meester, R., Penrose, M.D., Sarkar, A.: The random connection model in high dimensions. Stat. Probab. Lett. **35**(2), 145–153 (1997). https://doi.org/10.1016/S0167-7152(97)00008-4. https://www.sciencedirect.com/science/article/pii/S0167715297000084

30. Newman, C.M., Schulman, L.S.: One-dimensional $1/|j-i|^s$ percolation models: the existence of a transition for $s \leq 2$. Comm. Math. Phys. **104**(4), 547–571 (1986). http://projecteuclid.org/euclid.cmp/1104115167
31. Penrose, M.D.: Connectivity of soft random geometric graphs. Ann. Appl. Probab. **26**(2), 986–1028 (2016). https://doi.org/10.1214/15-AAP1110
32. Schulman, L.S.: Long range percolation in one dimension. J. Phys. A **16**(17), L639–L641 (1983). http://stacks.iop.org/0305-4470/16/639
33. Yukich, J.E.: Ultra-small scale-free geometric networks. J. Appl. Probab. **43**(3), 665–677 (2006). https://doi.org/10.1239/jap/1158784937